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The major results of this dissertation are theorems to the effect that certain classes of relational structures are not axiomatizable by universal sentences. Some of the particular classes considered are theories of measurement in the sense of the Scott-Suppes definition while others are theories of measurement according to a natural generalization of the above definition. Part of the significance of the results is that they are closely related to problems of proving representation theorems in measurement theory. Ideally, one would like to have a finite list of universal axioms which are both necessary and sufficient for guaranteeing the particular representation in which one is interested. The results of this technical report show that in many cases we are forced to settle for more modest achievements. Some intuitive statements of results whose precise formulations appear in the thesis are presented on (1) Additive Conjoint Measurement, (2) First-Order Segment of Decision Theory, (3) Difference Systems of Measurement, and (4) Multidimensional Scaling. (RP)

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SOME MODEL-THEORETIC RESULTS IN MEASUREMENT THEORY

BY

ROBERT JAY TITIEV

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PREFACE

The major results of this dissertation are theorems to the effect that certain classes of relational structures are not axiomatizable by universal sentences. Some of the particular classes considered are theories of measurement in the sense of the Scott-Suppes definition; others are theories of measurement according to a natural generalization of this definition. Part of the significance of the results is that they are closely related to problems of proving representation theorems in measurement theory. Ideally, one would like to have a finite list of universal axioms which are both necessary and sufficient for guaranteeing the particular representation in which one is interested. The results of this thesis show that in many cases we are forced to settle for more modest achievements.

What follows under the four headings below are some intuitive statements of results whose precise formulations appear in the thesis.

(1) On Additive Conjoint Measurement

Given a relation $\langle x_1, \ldots, x_n \rangle \geq \langle y_1, \ldots, y_n \rangle$, one cannot find a finite list of universal axioms which are both necessary and sufficient to guarantee a representation by real-valued functions, φ_i, i=1, ..., n, such that, for all appropriate $x_1, \ldots, x_n, y_1, \ldots, y_n$,

$$\langle x_1, \ldots, x_n \rangle \geq \langle y_1, \ldots, y_n \rangle \longleftrightarrow \sum_{i=1}^{n} \varphi_i(x_i) \geq \sum_{i=1}^{n} \varphi_i(y_i)$$

(2) On a First-Order Segment of Decision Theory

Let $\mathbf{C}$ be a set of consequences, and let $\mathbf{S} = \{s_1, \ldots, s_n\}$ be a finite list of states. Let acts be viewed as n-tuples of elements from $\mathbf{C}$. Then, on a preference relation, R, between acts, there can be no finite universal axiomatization which is both necessary and sufficient for the existence of a probability function p on $\mathbf{S}$ and a utility function u on $\mathbf{C}$ such that, for all $c_1, \ldots, c_n$, $c'_1, \ldots, c'_n \in \mathbf{C}$,

$$\langle c_1, \ldots, c_n \rangle \; R \; \langle c'_1, \ldots, c'_n \rangle \longleftrightarrow \sum_{i=1}^{n} u(c_i)\, p(s_i) \geq \sum_{i=1}^{n} u(c'_i)\, p(s_i)$$

(3) On Difference Systems of Measurement

Let D be a four-place relation on a set A. Let I be the binary relation (abDba $\wedge$ baDab). Let τ be the axiom

$$(\exists a,b,c \in A)\,[\neg(aIb) \wedge \neg(aIc) \wedge \neg(bIc) \rightarrow$$

$$(\exists a,b,c,d \in A)\,[\neg(aIb) \wedge \neg(aIc) \wedge abDcd \wedge cdDab] \,.$$

Then τ is a necessary axiom for measurement on an interval scale. Moreover, no finite list of universal axioms may be added to τ in order to obtain necessary and sufficient conditions for measurement on an interval scale.

(4) On Multidimensional Scaling

Let D be a four-place relation on a set A . Let ρ be a metric in Euclidean n-space. In multidimensional scaling one is interested in representations by vector-valued functions f such that, for all a, b, c, d$\in$A ,

$$abDcd \longleftrightarrow \rho(f(a), f(b)) \leq \rho(f(c), f(d)) .$$

For both the "dominance" metric and the ordinary Euclidean metric there can be no finite universal axiomatization which is necessary and sufficient for the above representation.

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CHAPTER I

PRELIMINARIES

1. Notation and Basic Concepts

Let's begin to immerse ourselves in the needed notions of logic, set theory and model theory by coming to terms with notation. Here is a brief beginning list:

'iff'	for	'if, and only if'
'st'	for	'such that'
'Dom(f)'	for	'the domain of the function f'
'Rng(f)'	for	'the range of the function f'
'A^1'	for	'A'
'A^{n+1}'	for	'$A \times A^n$'
'$\mathcal{P}(A)$'	for	'the power set of A'
'$\mathbb{R}$'	for	'the set of real numbers'
'$R_{\restriction A}$'	for	'the relation R, restricted to the set A'
'(x,y)'	for	'the interval $\{t \in \mathbb{R} / x < t < y\}$'
'$\langle x,y \rangle$'	for	'the ordered pair $\{\{x\}, \{x,y\}\}$'
'$\langle x_1, \ldots, x_n \rangle$'	for	'$\langle x_1, \langle x_2, \ldots, x_n \rangle\rangle$'
'$\Pi i,\ 1 \leq i \leq n$'	for	'the projection functions from $\mathbb{R}^n$ into $\mathbb{R}$ given by $\Pi i(\langle x_1, \ldots, x_n \rangle) = x_i$'

We shall be dealing with relational structures of the form $\mathfrak{M} = \langle A, R_1, \ldots, R_k \rangle$, where A is a non-empty set and $R_1, \ldots, R_k$ are relations on A of orders $m_1, \ldots, m_k$, respectively. The set A is the domain of the relational structure $\mathfrak{M}$, and the sequence $\langle m_1, \ldots, m_k \rangle$ is the type of $\mathfrak{M}$. We shall let '$\mathcal{D}(\mathfrak{M})$' denote the domain of $\mathfrak{M}$; '$|\mathfrak{M}|$' will refer to the cardinality of the domain of the relational structure $\mathfrak{M}$. If σ is a sentence of first-order logic, then 'σ tr $\mathfrak{M}$' will indicate that σ is true in the structure $\mathfrak{M}$. The homomorphisms in Tarski [15] are not the same as those in Scott-Suppes [9]. We shall deal with homomorphisms in the sense of the latter paper.

Definition

Let $\mathfrak{M} = \langle A, R_1, \ldots, R_k \rangle$ and $\mathfrak{N} = \langle B, S_1, \ldots, S_k \rangle$ be two relational structures of type $\langle m_1, \ldots, m_k \rangle$. Then a function f is a homomorphism from $\mathfrak{M}$ onto $\mathfrak{N}$ iff

(i) $\mathrm{Dom}(f) = A$, $\mathrm{Rng}(f) = B$, and

(ii) For all $i \in \{1, \ldots, k\}$ and all $\langle x_1, \ldots, x_{m_i} \rangle \in A^{m_i}$,

$\langle x_1, \ldots, x_{m_i} \rangle \in R_i$ iff $\langle f(x_1), \ldots, f(x_{m_i}) \rangle \in S_i$.

If also,

(iii) f is $1-1$,

then f is an isomorphism between $\mathfrak{M}$ and $\mathfrak{N}$.

If K is a class of relational structures, all of which are of a fixed type, then we shall follow Tarski [15] and let I(K) be the class of all isomorphic images of members of K . We shall let H(K) be the class of all structures $\mathfrak{M}$ for which there exists a structure $\mathfrak{N} \in K$ and a homomorphism h from $\mathfrak{N}$ to $\mathfrak{M}$. Similarly, $H^{-1}(K)$ will be the class of all structures $\mathfrak{M}$ for which there exists a structure $\mathfrak{N} \in K$ and a homomorphism h from $\mathfrak{M}$ to $\mathfrak{N}$. Finally, 'S(K)' will denote the class of all substructures of members of K . Where K is a unit class $\{\mathfrak{M}\}$, we shall write 'I($\mathfrak{M}$)' instead of 'I($\{\mathfrak{M}\}$)' , etc. The structure $\mathfrak{M}$ is <u>embeddable</u> in $\mathfrak{N}$ iff $\mathfrak{M} \in H^{-1}S(\mathfrak{N})$.

Now we are ready to start dealing with theories of measurement.

<u>Definitions</u> (Scott-Suppes)

A relational structure $\mathfrak{N}$ is a <u>numerical relational structure</u> iff $\mathcal{D}(\mathfrak{N}) = \mathbb{R}$. Let K be a class of relational structures of type $\langle m_1, \ldots, m_k \rangle$. Then K is a <u>theory of measurement</u> iff $I(K) \subseteq K$ and there exists a numerical relational structure $\mathfrak{N}$ st $K \subseteq H^{-1} S(\mathfrak{N})$.

Note that if $\mathfrak{N}$ is a numerical relational structure and $K = H^{-1} S(\mathfrak{N})$, then K is a theory of measurement which is closed under substructures. That is, $I(K) \subseteq K$ and $S(K) \subseteq K$.

Now that we have so many definitions at hand, perhaps the reader would like to see a proof of something. In Scott-Suppes [9], p. 116, there is the remark that "the class of <u>all</u> countable relational systems of a given type is a theory of measurement; however, the numerical relational system required is so bizarre as to be of no practical value." As a little exercise in set-theoretical methods of proof, let's consider an impractical generalization of the above statement.

Remark

Let ω_α be a cardinal number. Let K be a class of relational structures of type $\langle m_1, \ldots, m_k\rangle$ st, for every $\mathfrak{M}\epsilon K$, $|\mathfrak{M}| \leq \omega_\alpha$. Then there is a relational structure $\mathfrak{N}$ such that $|\mathfrak{N}| \leq 2^{\omega_\alpha}$ and $K \subseteq IS(\mathfrak{N})$.

Proof: We may assume $\mathfrak{M}\epsilon K \rightarrow \mathcal{D}(\mathfrak{M}) \subseteq \omega_\alpha$. By this assumption K is a set and, hence, can be indexed by an ordinal α^* of cardinality $\leq 2^{\omega_\alpha}$. Thus, let $K = \{\mathfrak{M}_\gamma | \gamma\epsilon\alpha^*\}$. Introduce distinct elements a^γ_τ for $\gamma\epsilon\alpha^*$, $\tau\epsilon\omega_\alpha$. Define $\bar{R}_h$ as follows, where $\mathfrak{M}_\gamma = \langle A_\gamma, R^\gamma_1, \ldots, R^\gamma_k\rangle$:
$\overline{R_h} = \{\langle a^\gamma_{i_1}, \ldots, a^\gamma_{i_{m_h}}\rangle / \langle i_1, \ldots, i_{m_h}\rangle \epsilon R^\gamma_h \text{ and } \gamma\epsilon\alpha^*\}$.
Let $\bar{A} = \{a^\gamma_\tau | \gamma\epsilon\alpha^*, \tau\epsilon\omega_\alpha\}$, $\mathfrak{N} = \langle \bar{A}, \bar{R}_1, \ldots, \overline{R_k}\rangle$. We shall now show that $K \subseteq IS(\mathfrak{N})$. Pick any $\mathfrak{M}_\gamma\epsilon K$. Let $A^* = \{a^\gamma_\delta / \delta\epsilon A_\gamma\}$, $\mathfrak{N}^* = \langle A^*, \bar{R}_{1\restriction A^*}, \ldots, \bar{R}_{k\restriction A^*}\rangle$. Then the map $\delta \rightarrow a^\gamma_\delta$ is an isomorphism between $\mathfrak{M}_\gamma$ and $\mathfrak{N}^*$. Hence, $\mathfrak{M}_\gamma \epsilon IS(\mathfrak{N})$.

qed.

2. Axiomatizability of Theories of Measurement

Let K be a class of relational structures all of the same type.

Let Σ be a set of sentences in first-order logic.

Definition (Tarski)

$K \in AC_\Delta$ [K is axiomatizable (in the extended sense)] iff there is a set Σ of sentences such that for all models $\mathfrak{M}$ (of the appropriate type)

$$\mathfrak{M} \in K \longleftrightarrow \Sigma \text{ tr } \mathfrak{M}$$

To indicate that K is finitely axiomatizable, or, in other words, that there is a unit set Σ as in the above definition, we shall write '$K \in AC$' . The notion of universal axiomatizability (in the extended sense), denoted by '$K \in \breve{U}C_\Delta$' , is obtained from the above definition by stipulating that Σ be a set of universal sentences. Finally, $K \in \breve{U}C$ is definable in the obvious way. Tarski [15] has elegant criteria characterizing classes in $\breve{U}C_\Delta$. In [16], Vaught develops a beautiful characterization of classes in $\breve{U}C$.

Since we shall almost always be dealing with measurement-theoretic classes whose members are finite relational structures, we need to use a slightly modified version of the apparatus set up by Tarski and Vaught. The following well-known result shows why.

Theorem 1

Let K be a class of finite relational structures of a fixed, finite type. Then the following are equivalent:

(i) $(\exists k \epsilon \omega)\ [\mathfrak{M} \epsilon K \rightarrow |\mathfrak{M}| < k] \wedge I(K) \subseteq K$

(ii) $K \epsilon AC$

(iii) $K \epsilon AC_\Delta$

So long as the cardinalities of the models in our theory of measurement K are unbounded, we know that $K \notin AC_\Delta$. Therefore, what we consider is axiomatizability in the following sense:

<u>Definition</u>

Let ω_α be a cardinal number. A class K of similar relational structures is <u>axiomatizable up to ω_α</u> $[K \epsilon AC(\omega_\alpha)]$ iff there is a sentence σ st , for all models $\mathfrak{M}$ (of the appropriate type) for which $|\mathfrak{M}| < \omega_\alpha$

$$\sigma \text{ tr } \mathfrak{M} \longleftrightarrow \mathfrak{M} \epsilon K$$

We define $K \epsilon UC(\omega_\alpha)$ by stipulating that σ be a universal sentence. Scott and Suppes mention in [9] that Vaught's characterization of classes in UC also works <u>mutatis mutandis</u> to provide a characterization of classes in $UC(\omega_0)$. In the following theorem to this effect we write '$S_n(\mathfrak{M})$' to indicate the class of all relational structures $\mathfrak{N}$ in $S(\mathfrak{M})$ such that $|\mathfrak{N}| < n$. We also write '$K(\omega_\alpha)$' to stand for 'the class of all models $\mathfrak{M} \epsilon K$ st $|\mathfrak{M}| < \omega_\alpha$'.

Theorem 2 (Vaught-Scott-Suppes)

Let K be a class of similar relational structures of finite order. Then $K \in UC(\omega_\alpha)$ iff

(i) $S(K(\omega_\alpha)) \subseteq K$

(ii) $I(K(\omega_\alpha)) \subseteq K$ and

(iii) $\exists n \in \omega$ st for all $\mathfrak{M}$ st $|\mathfrak{M}| < \omega_\alpha$,
if $S_{n+1}(\mathfrak{M}) \subseteq K$, then $\mathfrak{M} \in K$.

For the most part, we shall be interested in taking a fixed numerical relational structure $\mathfrak{N}$ and then considering the class K of all finite relational structures in $H^{-1}S(\mathfrak{N})$. Then we shall use Theorem 2 to show $K \notin UC(\omega_0)$. The reader can easily satisfy himself that, for each such result we obtain, there is the corresponding result that $H^{-1}S(\mathfrak{N}) \notin UC$. Since we shall be interested mainly in classes K such that $\mathfrak{M} \in K \rightarrow |\mathfrak{M}| < \omega_0$, it will be convenient for us to use the phrases 'K is axiomatizable' and 'K is universally axiomatizable' to mean $K \in AC(\omega_0)$ and $K \in UC(\omega_0)$, respectively.

3. Intuitive Comments

As Stevens says, "Measurement is possible only because there is a kind of isomorphism between (1) the empirical relations among properties of objects and events and (2) the properties of the formal game in which numerals are the pawns and operators the moves." ([10], pp. 20-21.) The formal definition of a theory of measurement given by Scott and Suppes is an attempt to render mathematically precise comments such as Stevens'. If $K \subseteq H^{-1}S(\mathfrak{n})$ is a theory of measurement, then the members of K reflect part (1) of Stevens' statement and the numerical relational structure $\mathfrak{n}$ reflects part (2).

A much disputed subject has been that of the measurement of subjective entities. Insofar as the extreme viewpoint that there can be no meaningful measurement of perceptions is concerned we have very convincing refutations available right at our fingertips thanks to psychophysical results such as those dealing with cross-modality matching (see Stevens [11]). One can, for example, under suitable laboratory conditions, make measurements of and reasonably accurate predictions about subjects' perceptions of the strengths of different vibrations applied to their fingertips.

In [13] Suppes and Zinnes attempt to provide theoretical criteria for the existence of measurement in general -- be it objective or subjective. So that we may have something specific to talk about in discussing some of the intuitive aspects of their formal approach, let us consider the four-place relation Δ on the real numbers given by $xy\Delta zw$ iff $x-y \leq z-w$. Let $\mathfrak{n}$ be the numerical relational structure $\langle \mathbb{R}, \Delta \rangle$. The intuitive idea of a homomorphism h from an empirical relational structure $\langle A, R \rangle$, $R \subseteq A^4$ into $S(\mathfrak{n})$ is that h provides a means of assigning numbers to the entities that are described in the structure $\langle A, R \rangle$. A paradigm that one might keep in mind is the case where A is a set of tones and the relation R is viewed as holding for a quadruple $\langle a, b, c, d\rangle$ of tones in A iff a particular individual has judged the difference in loudness between tones a and b to be not greater than the difference in loudness between tones c and d .

A representation theorem (see [13], pp. 4-8) for a theory of measurement $K \subseteq H^{-1}S(\mathfrak{n})$ is a result to the effect that certain axioms suffice to guarantee the membership of a relational structure in K . In our paradigm the intuitive idea behind such axioms is that if they are satisfied by an individual making judgments about tones, then we have a means of quantifying his perceptions. Thus, part of the Suppes-Zinnes criteria for meaningful measurement is that there be a representation theorem with axioms holding in empirical structures determined by experimental data. What is really desirable is the situation where an empirical structure satisfies axioms which are strong enough to guarantee one of several kinds of uniqueness results. Then the Suppes-Zinnes theory asserts that measurement has been achieved by means of a particular kind of scale (cf. Stevens [10], p. 25 and Suppes-Zinnes [13], pp. 8-15).

With respect to the particular case of the relation $x-y \leq z-w$ and the numerical relational structure $\mathfrak{n}$, nobody has, as yet, succeeded in finding a finite axiomatization that is necessary and sufficient for guaranteeing membership of finitary structures $\langle A, R\rangle$, $R \subseteq A^4$ in $H^{-1}S(\mathfrak{n})$. Sufficient but not necessary axioms appear in Suppes-Zinnes [13]. For an infinite necessary and sufficient universal axiomatization see Scott [8]. The question as to whether or not one can find a finite necessary and sufficient universal axiomatization has been answered negatively by Scott and Suppes [9]. In the next chapter we shall consider a proof of their theorem. Then we shall proceed to look at similar results for more general measurement-theoretic classes.

CHAPTER II

ESSENTIALLY ONE-DIMENSIONAL RESULTS

1. The Numerical Relation for Difference Systems of Measurement

We now turn to [9], where Scott and Suppes indicate a proof of the following result:

Theorem 3

Let Δ be the four-place relation on $\mathbb{R}$ given by $xy\Delta zw$ iff $x-y \leq z-w$. Let K be the class of all relational structures $\mathfrak{M} = \langle X, D\rangle$, where $D \subseteq X^4$, $|\mathfrak{M}| < \omega_0$, and $\mathfrak{M} \in H^{-1}S(\langle \mathbb{R}, \Delta\rangle)$. Then K is not axiomatizable by a universal sentence.

Our goal now is to establish the above theorem by using a lemma that will come in handy later.

Terminology:

Let m be an odd number. Let positive numbers $P_1, \ldots, P_m$ be given. Then the <u>a's generated by the P's</u> are the 2m numbers determined as follows: $a_1 = 1$. For $1 < j \leq m+1$, $a_j = a_{j-1} + P_{j-1}$. If $m+2 \leq j \leq 2m$ and j is odd, then $a_j = a_{j-1} + P(j-1)/2$. If $m+2 \leq j \leq 2m$ and j is even, then $a_j = a_{j-1} + P(j-m-1)/2$. Figure 1 pictures the a's generated by the P's.

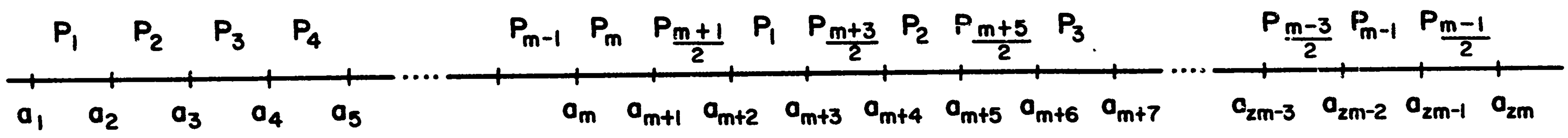

FIGURE 1

The a's Generated by the P's

Given any model $\mathfrak{M} = \langle A, D\rangle$, $D \subseteq A^4$ and any $a_1, \ldots, a_\ell \in A$, $\ell < |A|$, let us write '$\mathfrak{M}^{a_1, \ldots, a_\ell}$' to refer to the submodel $\langle A - \{a_1, \ldots, a_\ell\}, D\restriction_{A - \{a_1, \ldots, a_\ell\}}\rangle$ of $\mathfrak{M}$. We shall establish Theorem 3 by showing that K fails to satisfy the third condition of Theorem 2. To do this we need to show that, for all $n\in\omega$, there exists a model $\mathfrak{M}\notin K$ such that $|\mathfrak{M}| < \omega_0$ and $S_n(\mathfrak{M}) \subseteq K$. What we shall show is that, for all odd $m\in\omega$, $m \geq 3$, there exists a model $\mathfrak{M}\notin K$ st $|\mathfrak{M}| = 2m$ and, for all $a\in\mathfrak{D}(\mathfrak{M})$, $\mathfrak{M}^a \in K$. One can convince himself with a minor amount of effort that the above suffices to show K is not universally axiomatizable. The construction of the model $\mathfrak{M}$, given m, will depend upon selecting "nice" distances $P_1, \ldots, P_m$ and then using the a's generated by the P's as the elements of $\mathfrak{D}(\mathfrak{M})$. For the rest of the proof of Theorem 3, m will be a fixed odd integer ≥ 3.

In the following lemma, we shall be using distances $P_1, \ldots, P_m$ such that each P_i is a power of two and is larger than twice the sum of all the earlier P_i's. Because of the uniqueness property of binary expansions these conditions enable us to impose a strong limitation upon the number of equal-length relationships among intervals of the form (a_k, a_ℓ), where a_k and a_ℓ belong to the set of a's generated by the P's. An interval such as (a_k, a_ℓ) will be referred to as an <u>atomic interval</u> iff $a_\ell - a_k = P_i$, for some i, $1 \leq i \leq m$.

Lemma 1

Let $P_1, \ldots, P_m$ be powers of 2 such that if $1 \leq i \leq m - 1$ and $P_i = 2^j$, then $P_{i+1} \geq 2^{j+2}$. Let $A = \{a_1, \ldots, a_{2m}\}$ be the set of a's generated by the P's. Let $x, y, z, w \epsilon A$ such that $x > y$, $z > w$, $z > x$, and $x-y = z-w$. Then either

(i) (y, x) and (w, z) are atomic intervals or

(ii) (y, w) and (x, z) are atomic intervals or

(iii) $\langle y, x \rangle = \langle a_1, a_m \rangle$ and $\langle w, z \rangle = \langle a_{m+1}, a_{2m} \rangle$ or

(iv) $\langle y, x \rangle = \langle a_1, a_{m+1} \rangle$ and $\langle w, z \rangle = \langle a_m, a_{2m} \rangle$.

Proof

Let $\mathcal{L} = \{(t, u)/t, u \epsilon A, t < u \leq a_m\}$

$\mathcal{R} = \{(t, u)/t, u \epsilon A, a_{m+1} \leq t < u\}$

$\mathcal{S} = \{(t, u)/t, u \epsilon A, t \leq a_m, a_{m+1} \leq u\}$.

Then, for all $t, u \epsilon A$ st $t < u$, $(t, u) \epsilon \mathcal{L} \cup \mathcal{R} \cup \mathcal{S}$. Because $z > x$ and if $(t, u) \epsilon \mathcal{S}$, $(t', u') \epsilon \mathcal{L} \cup \mathcal{R}$, then $u-t > u'-t'$, we need only consider the following cases:

(1) $(y,x), (w,z) \epsilon \mathcal{L}$ or $(y,x), (w,z) \epsilon \mathcal{R}$.

(2) $(y,x) \epsilon \mathcal{L}$, $(w,z) \epsilon \mathcal{R}$.

(3) $(y,x), (w,z) \epsilon \mathcal{S}$.

We show first that case (1) is contradictory. It is clear from Figure 1 that no P_i can appear more than twice between any two points $a_i, a_j \epsilon A$. And, because x-y and z-w have identical binary expansions, we know that between x and y and between z and w there must be the same powers P_i occurring the same number of times. Hence, from case (1) we may conclude that $\langle x, y\rangle = \langle z, w\rangle$, and this contradicts the assumption $x < z$.

Case (2)

Let ℓ be the least integer such that P_ℓ appears between x and y in Figure 1. Let g be the greatest integer such that P_g appears between x and y in Figure 1. If $\ell = g$, then (y, x) and (w, z) are atomic intervals; qed. Hence, we may assume $\ell < g$. Then P_ℓ and $P_{\ell+1}$ must appear between x and y. Hence P_ℓ and $P_{\ell+1}$ must appear between z and w. If $1 \leq \ell < \frac{m-1}{2}$, then $P_{\frac{m+2\ell+1}{2}}$ occurs between P_ℓ and $P_{\ell+1}$ in (w, z). If $\frac{m+1}{2} \leq \ell \leq m-1$, then $P_{\ell-(\frac{m-1}{2})}$ occurs between P_ℓ and $P_{\ell+1}$ in (w, z); but this is a contradiction, given the definition of ℓ. Hence,

$g \geq \frac{m+2\ell+1}{2} \geq \frac{m+3}{2} > \frac{m+1}{2} > \ell$. This means $P_{\frac{m+3}{2}}$ and $P_{\frac{m+1}{2}}$ occur in (w, z). Therefore, P_1 occurs in (w, z) and $\ell = 1$. Similarly, $g = m - 1$. Therefore, $\langle y, x\rangle = \langle a_1, a_m\rangle$ and $\langle w, z\rangle = \langle a_{m+1}, a_{2m}\rangle$.

qed.

Case (3)

$y \leq a_m, a_{m+1} \leq x, w \leq a_m, a_{m+1} \leq z$. Also $z - w = x - y < z - y$. Hence $y < w$. Thus $(y, w) \in \mathcal{L}$, $(x, z) \in \mathcal{R}$ and $w - y = z - x$. So, as in case (2), either (y, w), (x, z) are atomic and we are done with the proof or else $\langle y, w\rangle = \langle a_1, a_m\rangle$ and $\langle x, z\rangle = \langle a_{m+1}, a_{2m}\rangle$. Then $\langle y, x\rangle = \langle a_1, a_{m+1}\rangle$ and $\langle w, z\rangle = \langle a_m, a_{2m}\rangle$.

qed.

We now proceed with the construction of the model $\mathfrak{M}$. Figure 2 shows why, for $m = 5$, $\mathfrak{M}$ fails to be a member of K. Any homomorphism from $\mathfrak{M}$ onto a subsystem of $\langle \mathbb{R}, \Delta \rangle$ must preserve the distances P_1, P_2, P_3, P_4 between a and b and also the distances P_3, P_1, P_4, P_2 between c and d. Therefore, in order that $\mathfrak{M}$ belong to K, the distance from a to b in $\mathfrak{M}$ must equal the distance from c to d in $\mathfrak{M}$. By constructing $\mathfrak{M}$ so that (a,b) is shorter than (c,d), we ensure that $\mathfrak{M} \notin K$.

Let A be as in Lemma 1. Let $a = a_1$, $b = a_m$, $c = a_{m+1}$, $d = a_{2m}$. Let $B_0 = \{\langle x,y,z,w \rangle \in A^4 / x - y < z - w\}$. Let $B_1 = \{\langle x,y,z,w \rangle \in A^4 / x - y = z - w$ and $\langle x,y,z,w \rangle$ is not a permutation of $\langle a,b,c,d \rangle\}$. Let $B_2 = \{\langle b,a,d,c \rangle, \langle b,d,a,c \rangle, \langle c,d,a,b \rangle, \langle c,a,d,b \rangle\}$. Let $D = B_0 \cup B_1 \cup B_2$ and take $\mathfrak{M} = \langle A,D \rangle$.

Claim: $\mathfrak{M} \notin K$

Proof by contradiction. Suppose there is a homomorphism $f: A \to \mathbb{R}$ st, for all $x,y,z,w \in A$, $\langle x,y,z,w \rangle \in D$ iff $f(x) - f(y) \leq f(z) - f(w)$. Then

$$f(a_2) - f(a_1) = f(a_{m+3}) - f(a_{m+2})$$
$$f(a_3) - f(a_2) = f(a_{m+5}) - f(a_{m+4})$$
$$f(a_4) - f(a_3) = f(a_{m+7}) - f(a_{m+6})$$
$$\vdots$$
$$f(a_m) - f(a_{m-1}) = f(a_{2m-1}) - f(a_{2m-2}).$$

Hence, $f(a_m) - f(a_1) = f(a_{2m}) - f(a_{m+1})$. Therefore, $\langle d,c,b,a \rangle \in D$.

* Contradiction. qed.

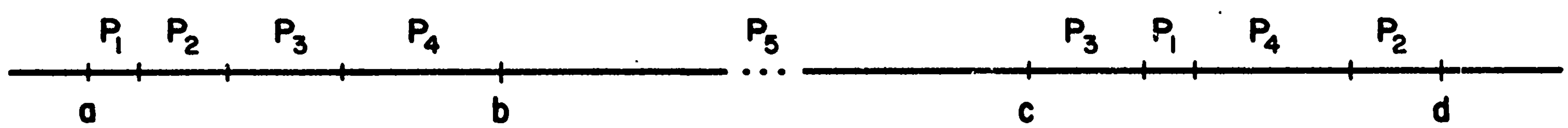

FIGURE 2

For the special case $m = 5$ shown above, $\mathfrak{m} \notin K$ because, as in the general case, length (a,b) in $\mathfrak{m}$ is less than length (c,d) in $\mathfrak{m}$

Claim: For all $a_i \epsilon A$, $\mathfrak{M}^{a_i} \epsilon K$.

Proof: Pick $\varepsilon > 0$ such that

$$2\varepsilon < \min \{(z - w) - (x - y) / \langle x,y,z,w\rangle \epsilon B_0\}$$

We shall define a homomorphism

$f\colon A - \{a_i\} \to \mathbb{R}$ according to which of the three following cases holds:

(i) $i \epsilon \{1, m, m+1, 2m\}$.

Then, for all $x,y,z,w \epsilon A - \{a_i\}$, $\langle x,y,z,w\rangle \epsilon D \leftrightarrow x - y \leq z - w$. Hence the identity function f embeds $\mathfrak{M}^{a_i}$.

qed.

(ii) $1 < i < m$. Then define f by

$$f(a_j) = \begin{cases} a_j + \varepsilon , & \text{if } j < i \\ a_j , & \text{if } j > i \end{cases}$$

Suppose $x,y,z,w \epsilon A - \{a_i\}$, and $\langle x,y,z,w\rangle \epsilon D$. We shall show that $f(x) - f(y) \leq f(z) - f(w)$. Note first that, if $\langle x,y,z,w\rangle \epsilon B_0$, then $f(x) - f(y) \leq f(z) - f(w)$, because

$$z - w \leq f(z) - [f(w) - \varepsilon]$$
$$\to z - w - \varepsilon \leq f(z) - f(w)$$

Hence, if $\langle x,y,z,w\rangle \epsilon B_0$,

$$f(x) - f(y) \leq x + \varepsilon - y < z - w - \varepsilon \leq f(z) - f(w) .$$

Suppose now that $\langle x,y,z,w\rangle \in B_1$.

Then $x - y = z - w$. We may assume that $x - y > 0$, so $x > y$ and $z > w$. If $z = x$, then $f(x) - f(y) = f(z) - f(w)$. Thus, we may assume $z > x$. Then, by Lemma 1, we may conclude that $f(x) - f(y) = f(z) - f(w)$, because, for any atomic interval (t,u) , where $t, u \notin \{a_i\}$, $f(t) - f(u) = t - u$.

Finally, suppose that $\langle x,y,z,w\rangle \in B_2$. To show that $f(x) - f(y) \leq f(z) - f(w)$ it suffices for us to verify that $f(b) - f(a) \leq f(d) - f(c)$. By the definition of f ,

$$f(b) - f(a) = b - (a + \epsilon) < b - a = d - c = f(d) - f(c) .$$

Hence we have completed the proof that, for all $x,y,z,w \in A - \{a_i\}$,

$$\text{if } \langle x,y,z,w\rangle \in D \text{ , then } f(x) - f(y) \leq f(z) - f(w) .$$

Now suppose that $x,y,z,w \in A - \{a_i\}$ and $f(x) - f(y) \leq f(z) - f(w)$. We must show $\langle x,y,z,w\rangle \in D$. If $z - w < x - y$, then, as above, $f(z) - f(w) < f(x) - f(y)$, which is a contradiction. Hence, $x - y \leq z - w$. If $x - y < z - w$, then $\langle x,y,z,w\rangle \in D$. qed.

Thus, we may suppose that $x - y = z - w$. If $\langle x,y,z,w\rangle$ is not a permutation of $\langle a,b,c,d\rangle$, then $\langle x,y,z,w\rangle \in B_1 \subseteq D$. qed. Hence, we may assume that $\langle x,y,z,w\rangle$ is a permutation of $\langle a,b,c,d\rangle$.

Since $x - y = z - w$,

either $\langle x,y,z,w\rangle \in \{\langle b,d,a,c\rangle, \langle b,a,d,c\rangle, \langle c,d,a,b\rangle, \langle c,a,d,b\rangle\}$

or $\langle x,y,z,w\rangle \in \{\langle a,b,c,d\rangle, \langle a,c,b,d\rangle, \langle d,b,c,a\rangle, \langle d,c,b,a\rangle\}$.

The latter case is ruled out, since $f(a) - f(b) > f(c) - f(d)$. Hence $\langle x,y,z,w\rangle \in B_2 \subseteq D$. qed.

(iii) $m < i$ and $i \notin \{m+1, 2m\}$

In this case we define f by

$$f(a_j) = \begin{cases} a_j & , \text{ if } j < i \\ a_j + \epsilon, & \text{ if } j > i \end{cases}$$

Then, as was done above, one can verify that, for all $x,y,z,w \in A - \{a_i\}$,

$$\langle x,y,z,w\rangle \in D \quad \text{iff} \quad f(x) - f(y) \leq f(z) - f(w) .$$

Hence we have completed the proof of Theorem 3. Let us now turn to some related results pertaining to measurement theory and decision theory.

We shall consider a well-known subclass of the structures of the form $\mathcal{B} = \langle A,p\rangle$, where A is a non-void set and $p: A^2 \to \mathbb{R}$.

Definition (Suppes-Zinnes [13], pp. 48-49)

$\mathcal{B} = \langle A,p\rangle$ is a B.T.L. (Bradley-Terry-Luce) system iff, for all $a,b,c \in A$,

(i) $0 < P_{ab}$

(ii) $P_{ab} + P_{ba} = 1$, and

(iii) $\dfrac{P_{ab}}{P_{ba}} \cdot \dfrac{P_{bc}}{P_{cb}} = \dfrac{P_{ac}}{P_{ca}}$

Corollary 1

Let K be the class of structures $\langle A,D\rangle$ such that $A \neq \emptyset$, $|A| < \omega_0$, $D \subseteq A^4$, and there exists a B.T.L. system $\mathcal{B} = \langle A,p\rangle$ such that, for all $x,y,z,w \in A$,

$$xyDzw \quad \text{iff} \quad P_{xy} \leq P_{zw}$$

Then K is not axiomatizable by a universal sentence.

2. A Corollary about Additive Conjoint Measurement

In Chapter 6 of a forthcoming work by Krantz, Luce, Suppes, and Tversky [2], tentatively titled Foundations of Measurement, structures of the form $\langle A_1, \ldots, A_n, \geq \rangle$ are considered, where $A_1, \ldots, A_n$ are non-void sets and $\geq$ is a binary relation on $A_1 x \ldots x A_n$. A five axiom representation theorem is given providing sufficient conditions for the existence of real-valued functions φ_i on A_i, $i = 1, \ldots, n$, such that

(1) For all $p_i, q_i \in A_i$,

$$\langle p_1, \ldots, p_n\rangle \geq \langle q_1, \ldots, q_n\rangle \text{ iff } \sum_{i=1}^{n} \varphi_i(p_i) \geq \sum_{i=1}^{n} \varphi_i(q_i) .$$

Let us now look at the kind of first-order language we shall need in order to deal with questions of axiomatizability concerning structures of the form $\langle A_1, \ldots, A_n, \geq \rangle$. We consider a language $\mathcal{L}$ in which there are n unary relation symbols R_i, $i = 1, \ldots, n$, and one $(2n)$-ary relation symbol S. Each R_i corresponds to the membership relation for the set A_i, and S corresponds to the relation $\geq$. We now shall use Theorem 3 to establish:

Corollary 2

Let K be the class of all finite models $\langle A, A_1, \ldots, A_n, B\rangle$ in $\mathcal{L}$ such that if $A_1, \ldots, A_n$ are non-void, then there exist representing functions $\varphi_i : A_i \to \mathbb{R}$ as in (1). Then K is not axiomatizable by a universal sentence.

Proof:

As before, given an odd $m \geq 3$, we need only find a model $\mathfrak{M} = \langle A, A_1, \ldots, A_n, B\rangle$ such that $|A| = 2m$, $\mathfrak{M} \notin K$, but, for all $a \in A$, $\mathfrak{M}^a \in K$. Let A be as in Lemma 1. Let $D \subseteq A^4$ be as in the proof of Theorem 3. Let $A_1 = A_2 = A$, $A_i = \{a_1\}$, $3 \leq i \leq n$. Define the relation $B \subseteq A^{2n}$ by

$$\langle p_1, \ldots, p_n, q_1, \ldots, q_n\rangle \in B \quad \text{iff} \quad \langle p_1, q_1, q_2, p_2\rangle \in D .$$

Now let $\mathfrak{M} = \langle A, A_1, \ldots, A_n, B\rangle$.

Claim: $\mathfrak{M} \notin K$.

Suppose the contrary. Then there are functions φ_i as in (1). Hence,

(2) For all $p_1, p_2, q_1, q_2 \in A$,

$$\langle p_1, q_1, q_2, p_2\rangle \in D \quad \text{iff} \quad \varphi_1(p_1) + \varphi_2(p_2) \geq \varphi_1(q_1) + \varphi_2(q_2) .$$

Pick any $a_k, a_\ell \in A$.

Since $\langle a_k, a_\ell, a_k, a_\ell \rangle \in D$,

$$\varphi_1(a_k) + \varphi_2(a_\ell) \geq \varphi_1(a_\ell) + \varphi_2(a_k) .$$

That is,

$$\varphi_1(a_k) - \varphi_1(a_\ell) \geq \varphi_2(a_k) - \varphi_2(a_\ell) .$$

Also, $\langle a_\ell, a_k, a_\ell, a_k \rangle \in D$. Hence

$$\varphi_1(a_\ell) + \varphi_2(a_k) \geq \varphi_1(a_k) + \varphi_2(a_\ell) .$$

Thus, $\varphi_2(a_k) - \varphi_2(a_\ell) \geq \varphi_1(a_k) - \varphi_1(a_\ell)$.

Hence, $\varphi_1(a_k) - \varphi_1(a_\ell) = \varphi_2(a_k) - \varphi_2(a_\ell)$,

for all $1 \leq k, \ell \leq 2m$.

In particular,

$$\varphi_2(a_k) = \varphi_1(a_k) + \varphi_2(a_1) - \varphi_1(a_1) ,$$

for all $1 \leq k \leq 2m$.

Therefore, by (2),

For all $p_1, p_2, q_1, q_2 \in A$

$$\langle p_1, q_1, q_2, p_2 \rangle \in D \quad \text{iff} \quad \varphi_1(p_1) + \varphi_1(p_2) \geq \varphi_1(q_1) + \varphi_1(q_2) .$$

But then the function $f = -\varphi_1$ homomorphically embeds $\langle A,D \rangle$ in $\langle \mathbb{R},\Delta \rangle$, which is a contradiction. Hence, $\mathfrak{M} \notin K$. qed.

Now, by establishing the following claim, we shall complete the proof of Corollary 2.

Claim: For all $a \epsilon A$, $\mathfrak{M}^a \epsilon K$.

Since $\mathfrak{M}^{a_1} \epsilon K$, we may assume $a \neq a_1$. We know that there is a function f embedding $\langle A - \{a\}, D_{\restriction A - \{a\}} \rangle$ in $\langle \mathbb{R}, \Delta \rangle$.

Let $\varphi_1 = \varphi_2 = -f$, $\varphi_i \equiv 0$, all $3 \leq i \leq n$.

Then, for all $p_i, q_i \epsilon A - \{a\}$,

$$\langle p_1, \ldots, p_n, q_1, \ldots, q_n \rangle \epsilon B \quad \text{iff} \quad \sum_{i=1}^{n} \varphi_i(p_i) \geq \sum_{i=1}^{n} \varphi_i(q_i) .$$

Hence, $\mathfrak{M}^a \epsilon K$. qed.

3. A Corollary Pertaining to Decision Theory

We shall now use Theorem 3 to prove a non-axiomatizability result related to decision theory. Our framework will be a first-order rendering of only a small portion of the theory developed in Savage [7]. Let us now consider the version of Savage's representation theorem which is stated in Luce-Raiffa [3], pp. 300-304. A collection $\mathfrak{G}$ of acts is given; members of $\mathfrak{G}$ are functions defined on a set $\mathfrak{S}$ of states and having values in a set $\mathfrak{C}$ of consequences. Events are subsets E of $\mathfrak{S}$. There is a binary relation $\geq$ of preference between acts. Savage's representation theorem provides powerful second-order axioms guaranteeing the existence of a probability function p defined on the events and a utility function u defined on the consequences such that the following holds:

(3) Let $\{E_i\}$ $1 \leq i \leq m$ and $\{E'_j\}$ $1 \leq j \leq n$ be two partitions of $\mathfrak{S}$. Let A, A' be acts such that $A(s) = c_i$, for all $s \epsilon E_i$ and $A'(s) = c'_j$,

for all $s \epsilon E'_j$. Then

$$A \geq A' \quad \text{iff} \quad \sum_{i=1}^{m} u(c_i)\, p(E_i) \geq \sum_{j=1}^{n} u(c'_j)\, p(E'_j) .$$

Suppose now that we consider only a finite set of states $\mathcal{S} = \{s_1, \ldots, s_n\}$. Each act may then be viewed as an n-tuple $\langle c_1, \ldots, c_n \rangle$ of consequences. We might wonder about the question of finding axioms on a binary relation $\geq$ between acts such that functions p and u exist satisfying (3). Then we would have:

(4) For all $c_1, \ldots, c_n, c'_1, \ldots, c'_n \epsilon \mathcal{C}$

$$\langle c_1, \ldots, c_n \rangle \geq \langle c'_1, \ldots, c'_n \rangle \quad \text{iff} \quad \sum_{i=1}^{n} u(c_i)p(s_i) \geq \sum_{i=1}^{n} u(c'_i)p(s_i) .$$

In proving that there is no first-order universal axiomatization giving necessary and sufficient conditions for the existence of a probability and a utility function satisfying (4), we lose no generality by dealing with the binary relation $\geq$ on $\mathcal{C}^n$ as a (2n)-ary relation on $\mathcal{C}$.

Corollary 3

Let $\mathcal{S} = \{s_1, \ldots, s_n\}$ be a fixed set of n states. Let K be the class of all finite models $\langle \mathcal{C}, \kappa \rangle$ such that $R \subseteq \mathcal{C}^{2n}$ and there exist real-valued functions u on $\mathcal{C}$ and p on $\mathcal{S}$ such that

(i) $p(s_i) \geq 0$, $i = 1, \ldots, n$

(ii) $\sum_{i=1}^{n} p(s_i) = 1$, and

(iii) For all $c_1, \ldots, c_n, c_1', \ldots, c_n' \in \mathbf{C}$,

$$\langle c_1, \ldots, c_n, c_1', \ldots, c_n' \rangle \in R \text{ iff } \sum_{i=1}^{n} u(c_i)p(s_i) \geq \sum_{i=1}^{n} u(c_i')p(s_i)$$

Then K is not axiomatizable by a universal sentence.

<u>Proof</u>:

Let A and D be as in the proof of Corollary 2. Let $R \subseteq A^{2n}$ be given by:

$$\langle p_1, \ldots, p_n, q_1, \ldots, q_n \rangle \in R \text{ iff } \langle p_1, q_1, q_2, p_2 \rangle \in D .$$

Let $\mathfrak{M} = \langle A,R \rangle$.

Claim: $\mathfrak{M} \notin K$.

Suppose the contrary. Then there exist functions p and u satisfying (i)-(iii). It follows that, for all $p_1, p_2, q_1, q_2 \in A$,

$$\langle p_1, q_1, q_2, p_2 \rangle \in D \quad \text{iff} \quad u(p_1)\, p(s_1) + u(p_2)\, p(s_2)$$
$$\geq u(q_1)\, p(s_1) + u(q_2)\, p(s_2)$$

Let $\varphi_i : A \to \mathbb{R}$ by $\varphi_i(a) = u(a)\, p(s_i)$, $i = 1, 2.$

Then, for all $p_1, p_2, q_1, q_2 \in A$,

$$\langle p_1, q_1, q_2, p_2\rangle \epsilon D \quad \text{iff} \quad \varphi_1(p_1) + \varphi_2(p_2) \geq \varphi_1(q_1) + \varphi_2(q_2) .$$

Thus, exactly as was done in the proof of Corollary 2, we may reach a contradiction.

Hence, $\mathfrak{M} \notin K$. qed.

Claim: For all $a \epsilon A$, $\mathfrak{M}^a \epsilon K$

Choose a homomorphism f from $\langle A - \{a\} , D_{\upharpoonright A - \{a\}}\rangle$ into $\langle \mathbb{R}, \Delta\rangle$.

Let $u = -f$. Define $p : \mathcal{S} \to \mathbb{R}$ by $p(s_1) = p(s_2) = \frac{1}{2}$, $p(s_j) = 0$, $3 \leq j \leq n$. Then, for all $p_1, \ldots, p_n, q_1, \ldots, q_n \epsilon A - \{a\}$,

$$\begin{aligned} \langle p_1, \ldots, p_n, q_1, \ldots, q_n\rangle \epsilon R \quad &\text{iff} \quad \langle p_1, q_1, q_2, p_2 > \epsilon D \\ &\text{iff} \quad f(p_1) - f(q_1) \leq f(q_2) - f(p_2) \\ &\text{iff} \quad u(p_1) + u(p_2) \geq u(q_1) + u(q_2) \\ &\text{iff} \quad \sum_{i=1}^{n} u(p_i)p(s_i) \geq \sum_{i=1}^{n} u(q_i)p(s_i) \end{aligned}$$

Hence, $\mathfrak{M}^a \epsilon K$. qed.

Thus, we have completed the proof of Corollary 3.

4. A Simple Representation Theorem with Universal Axioms

In the previous section we showed that within a certain framework for talking about a preference relation on acts one cannot find a finite list of universal axioms which give necessary and sufficient conditions that the ordering of acts be in accord with the principle of maximizing expected utility. Other criteria that have been proposed are discussed and characterized in Milnor [5]. We now mention that Wald's minimax criterion may be viewed in terms of a representation theorem. The following result states that there is a universal sentence which is a necessary and sufficient condition that a preference relation on acts be in accord with the Wald criterion. Note that, with respect to the class K below, the number of states of nature is fixed, although structures in K may have domains (sets of consequences) of arbitrary finite or countably infinite cardinalities. We need the condition that the number of states of nature be fixed in order to know that the members of K are of the same type. This condition also means that (i), (ii), and (iii) below are equivalent to a single universal axiom.

Theorem 4

Let n be a fixed positive integer ≥ 2. Let K be the class of all structures $\langle A, \leq \rangle$ such that A is a non-empty denumerable set, $\leq$ is a binary relation on A^n, and there exists a function $u : A \to \mathbb{R}$ such that, for all $c_1, \ldots, c_n, c'_1, \ldots, c'_n \in A$,

$$\langle c_1, \ldots, c_n \rangle \leq \langle c'_1, \ldots, c'_n \rangle \text{ iff } \min \{u(c_1), \ldots, u(c_n)\} \leq \min \{u(c'_1), \ldots, u(c'_n)\} .$$

Then K is axiomatizable (up to ω_1) by a universal sentence. Moreover, for a given structure in K, the function u is unique up to a monotone (non-decreasing) transformation.

Proof: It is straightforward to show that the following is a finite universal axiomatization for K .

(i) $\leq$ is transitive and connected

(ii) $\langle a_1, \ldots, a_1\rangle \leq \langle b_1, \ldots, b_1\rangle$

$$\wedge \langle a_1, \ldots, a_1\rangle \leq \langle a_2, \ldots, a_2\rangle \leq \ldots \leq \langle a_n, \ldots, a_n\rangle$$
$$\wedge \langle b_1, \ldots, b_1\rangle \leq \langle b_2, \ldots, b_2\rangle \leq \ldots \leq \langle b_n, \ldots, b_n\rangle$$
$$\rightarrow \langle a_1, a_2, \ldots, a_n\rangle \leq \langle b_1, b_2, \ldots, b_n\rangle$$

(iii) Let σ, τ be permutations of $\{1, 2, \ldots, n\}$. Then

$$\langle a_1, a_2, \ldots, a_n\rangle \leq \langle b_1, b_2, \ldots, b_n\rangle \rightarrow \langle a_{\sigma(1)}, \ldots, a_{\sigma(n)}\rangle \leq \langle b_{\tau(1)}, \ldots, b_{\tau(n)}\rangle .$$

5. A Numerical Dissimilarity Relation Imposed by the Absolute Value Metric

Some terminological clarification may be in order with regard to the above heading. The name 'similarity relation' is sometimes used to denote relations that are reflexive and symmetric. For this reason we have chosen to use the phrase 'dissimilarity relation' in order to refer to relations that are in some manner related to judgments about similarities and dissimilarities between objects. We shall now consider the dissimilarity relation $|x - y| \leq |z - w|$ on the real numbers. Our goal is to prove.

Theorem 5

Let Δ be the four-place relation on $\mathbb{R}$ given by $xy\Delta zw$ iff $|x - y| \leq |z - w|$. Let K be the class of all models $\mathfrak{M} = \langle A,D\rangle$ such that $|\mathfrak{M}| < \omega_0$, $D \subseteq A^4$, and $\mathfrak{M}$ is homomorphic to a substructure of $\langle \mathbb{R},\Delta\rangle$. Then K is not axiomatizable by a universal sentence.

Proof: As before, let $m \geq 3$ be an odd integer. Let A,a,b,c,d be as in the proof of Theorem 3. We shall construct a model $\mathfrak{M} = \langle A,D\rangle$ such that $\mathfrak{M} \notin K$, but, for all $x \in A$, $\mathfrak{M}^x \in K$.

Let $B_0 = \{\langle x,y,z,w\rangle \in A^4 / |x - y| < |z - w|\}$

$B_1 = \{\langle x,y,z,w\rangle \in A^4 / |x - y| = |z - w|$ and $\langle x,y,z,w\rangle$ is not a permutation of $\langle a,b,c,d\rangle\}$

$B_2 = \{\langle a,b,c,d\rangle, \langle a,b,d,c\rangle, \langle b,a,c,d\rangle, \langle b,a,d,c\rangle, \langle a,c,b,d\rangle, \langle a,c,d,b\rangle, \langle c,a,b,d\rangle, \langle c,a,d,b\rangle\}$.

Let $D = B_0 \cup B_1 \cup B_2$. Take $\mathfrak{M} = \langle A,D\rangle$.

Claim: $\mathfrak{M} \notin K$.

Suppose f embeds $\mathfrak{M}$ in $\langle \mathbb{R},\Delta\rangle$. We first show by induction that f must be strictly monotone on $\{a_1, \ldots, a_{2m}\}$. Since $\langle a_1, a_2, a_1, a_1\rangle \notin D$, we know $f(a_1) \neq f(a_2)$. Suppose $f(a_1) < f(a_2)$, $k \geq 3$, and $f(a_i) < f(a_{i+1})$, for $1 \leq i \leq k - 2$. We shall show that $f(a_{k-1}) < f(a_k)$. Suppose not. Then

$$f(a_1) \leq f(a_k) \Rightarrow 0 < f(a_k) - f(a_1) \leq f(a_{k-1}) - f(a_1)$$
$$\Rightarrow |f(a_k) - f(a_1)| \leq |f(a_{k-1}) - f(a_1)|$$
$$\Rightarrow \langle a_k, a_1, a_{k-1}, a_1\rangle \in D$$

* Contradiction

Also,

$$f(a_k) < f(a_1) \Rightarrow 0 < f(a_1) - f(a_k) \leq f(a_{k-1}) - f(a_k)$$
$$\Rightarrow \langle a_1, a_k, a_{k-1}, a_k\rangle \in D$$

* Contradiction

Hence, $f(a_{k-1}) < f(a_k)$. qed.

Similar reasoning shows that if $f(a_2) < f(a_1)$, then $f(a_{2m}) < f(a_{2m-1}) < \ldots < f(a_2) < f(a_1)$. But, because f is strictly monotone, one can show, as in the proof of Theorem 3, that we must have

$$|f(a_m) - f(a_1)| = |f(a_{2m}) - f(a_{m+1})|.$$

Therefore, $\langle c,d,a,b\rangle \in D$.

* Contradiction

Hence, $\mathfrak{M} \notin K$. qed.

<u>Claim:</u> For all $a_i \in A$, $\mathfrak{M}^{a_i} \in K$.

Let $\epsilon > 0$ be such that

$$2\epsilon < \min \{|z - w| - |x - y| / \langle x,y,z,w\rangle \in B_0\}$$

Define f on $A - \{a_i\}$ exactly as was done in the proof of Theorem 3. Then f is strictly increasing; moreover, for all $x,y,z,w \in A - \{a_i\}$,

$$\langle x,y,z,w\rangle \in B_0 \rightarrow |f(x) - f(y)| < |f(z) - f(w)|$$

and $\quad \langle x,y,z,w\rangle \in B_1 \rightarrow |f(x) - f(y)| = |f(z) - f(w)|$

Finally, one can check that $|f(a) - f(b)| < |f(c) - f(d)|$ and $|f(a) - f(c)| < |f(b) - f(d)|$. Therefore, for all $x,y,z,w \in A - \{a_i\}$,

$$\langle x,y,z,w\rangle \in D \rightarrow |f(x) - f(y)| \leq |f(z) - f(w)| .$$

Now suppose that $x,y,z,w \in A - \{a_i\}$ and $|f(x) - f(y)| \leq |f(z) - f(w)|$. We need only show that $\langle x,y,z,w\rangle \in D$. If $|z - w| < |x - y|$, then $\langle z,w,x,y\rangle \in B_0$ and, by the above, $|f(z) - f(w)| < |f(x) - f(y)|$, which is a contradiction. Hence, $|x - y| \leq |z - w|$. Suppose $\langle x,y,z,w\rangle \notin D$. We shall obtain a contradiction. By our supposition $|x - y| = |z - w|$ and $\langle x,y,z,w\rangle$ is a permutation of $\langle a,b,c,d\rangle$. Since $\langle x,y,z,w\rangle \notin B_2$, we may conclude that $\langle x,y,z,w\rangle \in \{\langle b,d,a,c\rangle, \langle b,d,c,a\rangle, \langle d,b,a,c\rangle, \langle d,b,c,a\rangle, \langle c,d,a,b\rangle, \langle c,d,b,a\rangle, \langle d,c,a,b\rangle, \langle d,c,b,a\rangle\}$. From this it follows that $|f(x) - f(y)| > |f(z) - f(w)|$.

* Contradiction

qed.

This completes the proof of Theorem 5. For convenience in deriving the higher dimensional results of Chapter III, we note the following consequence of the above proof.

<u>Lemma 2</u>

For all odd $m \geq 3$, there exist $2m$ real numbers $0 < a_1 < a_2 < \ldots < a_{2m}$ and a four-place relation D on

$X = \{a_1, \ldots, a_{2m}\}$ st the model $\mathfrak{M} = \langle X, D\rangle$ is not in K, yet, for all $\epsilon > 0$, every $2m - 1$ element submodel of $\mathfrak{M}$ is homomorphic to a substructure of $\langle \mathbb{R}, \Delta\rangle$ by a homomorphism f such that, for each $a_i \epsilon \mathcal{D}(f)$, $f(a_i) \epsilon [a_i, a_i + \epsilon)$.

6. A Corollary Pertaining to Utility Differences

Structures of the form $\langle A, Q, R\rangle$, where A is a non-void set, $Q \subseteq A^2$, and $R \subseteq A^4$ are considered in Suppes-Winet [12]. The set A has as its intended interpretation a set of alternatives; Q is interpreted as a preference relation on A and R as a relation holding between alternatives x,y,z,w iff the difference in preference between x and y fails to exceed the difference in preference between z and w. Suppes and Winet present an axiomatization sufficient for the existence on A of a representing utility function u which is unique up to a linear transformation. As is also the case in the von Neumann and Morgenstern approach to utility, the axioms are stated outside the framework of a first-order language. The following result rules out the possibility of a finite universal axiomatization within a first-order language.

Corollary 4

Let K be the class of structures $\langle A, R, Q\rangle$ such that A is a non-void finite set, $R \subseteq A^4$, $Q \subseteq A^2$ and there exists a function $u: A \to \mathbb{R}$ st, for all $x,y,z,w \,\epsilon A$,

(i) xQy iff $u(x) \geq u(y)$ and

(ii) $xyRzw$ iff $|u(x) - u(y)| \leq |u(z) - u(w)|$.

Then K is not universally axiomatizable.

Proof:

Let A, D be as in the proof of Theorem 5. Let $E \subseteq A^2$ be given by xEy iff $x \geq y$. Let $\mathfrak{M} = \langle A, D, E \rangle$. Then $\mathfrak{M} \notin K$; but, for all $a \in A$, $\mathfrak{M}^a \in K$. qed.

7. An Eight-Place Dissimilarity Relation

The intuitive background behind the next theorem is a special kind of conjoint measurement situation. We may imagine that a subject is considering various objects, each of which is split into two parts, and that he is making dissimilarity judgments on the basis of how much there is of some specific quality within each of the two parts of the objects under consideration.

Theorem 6

Let $\emptyset$ be the eight-place relation on the real numbers given by

$$x_1\, y_1\, x_2\, y_2\, \emptyset\, z_1\, z_2\, w_1\, w_2 \quad \text{iff}$$

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \leq \sqrt{(z_1 - z_2)^2 + (w_1 - w_2)^2} .$$

Let K be the class of models $\mathfrak{M} = \langle A, D \rangle$ such that $|\mathfrak{M}| < \omega_0$, $D \subseteq A^8$ and $\mathfrak{M} \in H^{-1} S(\langle \mathbb{R}, \emptyset \rangle)$. Then K is not axiomatizable by a universal sentence.

One can prove Theorem 6 by judiciously selecting powers of two for Lemma 1 so that if $\alpha_1, \alpha_2, \beta_1, \beta_2$ are any distances between pairs of a's generated by the P's and $\alpha_1^2 + \alpha_2^2 = \beta_1^2 + \beta_2^2$, then $\alpha_1 = \beta_1$ or $\alpha_1 = \beta_2$. Then one can proceed along the lines of the proof of Theorem 5.

CHAPTER III

GENERAL RESULTS IN n-DIMENSIONS

1. The Ordinary n-Dimensional Euclidean Metric

In this chapter we give up the idea of embeddings only into the real numbers and turn to questions of embeddings in $\mathbb{R}^n$. We shall be concerned with the dissimilarity relation Δ_n imposed by the Euclidean metric in $\mathbb{R}^n$, which we shall denote by '$\overline{}^{n}$', or simply by '$\overline{}$'.

Definitions

Let n be a positive integer.

Let $x = \langle x_1, \ldots, x_n\rangle$, $y = \langle y_1, \ldots, y_n\rangle \in \mathbb{R}^n$

$$\text{Then} \quad \overline{x,y}^{\,n} = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$$

Let Δ_n be the four-place relation on $\mathbb{R}^n$ given by $xy\Delta_n zw$ iff $\overline{x,y}^{\,n} \leq \overline{z,w}^{\,n}$. Let K_n be the class of all models $\mathfrak{M} = \langle X, D\rangle$, such that $|\mathfrak{M}| < \omega_0$, $D \subseteq X^4$, and $\mathfrak{M}$ is homomorphic to a substructure of $\langle \mathbb{R}^n, \Delta_n\rangle$.

We shall be after the result that none of the classes K_n, $n = 1, 2, \ldots$ is axiomatizable by a universal sentence.

The enterprise of multidimensional scaling involves the hope that one can measure "psychological distances" by using mathematical distances. One would like to represent objects as points in $\mathbb{R}^n$ so that experimental dissimilarities between the objects are reflected by the distances between the points representing the objects. An embedding in $\mathbb{R}^n$ may be thought of as having the meaning that there are n properties on the basis of which discriminations between the objects are being made. As an example, let us suppose that $\mathfrak{M}$ is an empirical relational structure stemming from experimental dissimilarity data. Suppose that h is a function which homomorphically embeds $\mathfrak{M}$ in the relational structure $\langle \mathbb{R}^n, \Delta_n \rangle$. Then we may think of h as indicating something to us about the relative importance of the n stimulus properties. An extreme situation would be the case where $\Pi_1(x)$ is constant for all $x \in \mathrm{Rng}(h)$; here we would have grounds for concluding that the first property of the stimuli had no influence upon the behavior of the subjects in the experiment.

Now let's turn away from psychological intuitions and consider some mathematical ones, <u>viz</u>., the geometric intuitions that lie behind the proof of Theorem 7 below. We want to show that the classes K_n fail to satisfy the third condition of Theorem 2. We shall build up models $\mathfrak{M}$ out of points such as $a_1, \ldots, a_{2m}$ in Lemma 2. Our aim will be to show that any homomorphism h from $\mathfrak{M}$ into $\langle \mathbb{R}^n, \Delta_n \rangle$ must be such that $h(a_1), \ldots, h(a_{2m})$ are collinear, because, given this, we may then use our one-dimensional results to prove that $\mathfrak{M} \notin K_n$. In order to build $\mathfrak{M}$ so that the collinearity property holds, we shall adjoin n elements $b_1, \ldots, b_n$ to the domain of the model for the one-dimensional proof and then we shall appropriately extend the four-place relation D to the new domain.

Figure 3 shows the ideas behind the models used in the proofs for K_2 and K_3. In the case of K_2 we adjoin points b_1 and b_2 to $a_1, \ldots, a_{2m}$ and extend the relation D by stipulating that each of the a_i's be equidistant from b_1 and b_2. This condition obviously forces the collinearity of $h(a_1), \ldots, h(a_{2m})$, for any homomorphism h. Similarly, in the case of K_3, we arrange things so that, in $\mathbb{R}^3$, $h(a_1), \ldots, h(a_{2m})$ must be on a line perpendicular to the plane of the equilateral triangle having vertices $h(b_1)$, $h(b_2)$, $h(b_3)$.

We now begin the proof of

<u>Theorem 7</u>

Let n be a positive integer and let K_n be as defined at the beginning of this chapter. Then K_n is not axiomatizable by a universal sentence.

The first thing we shall consider is the proof of an n-dimensional analogue to the theorem that the locus of points in a plane that are equidistant from two distinct points is a straight line.

<u>Lemma 3</u>

Let $q_1, \ldots, q_n$ be n distinct points in $\mathbb{R}^n$ such that

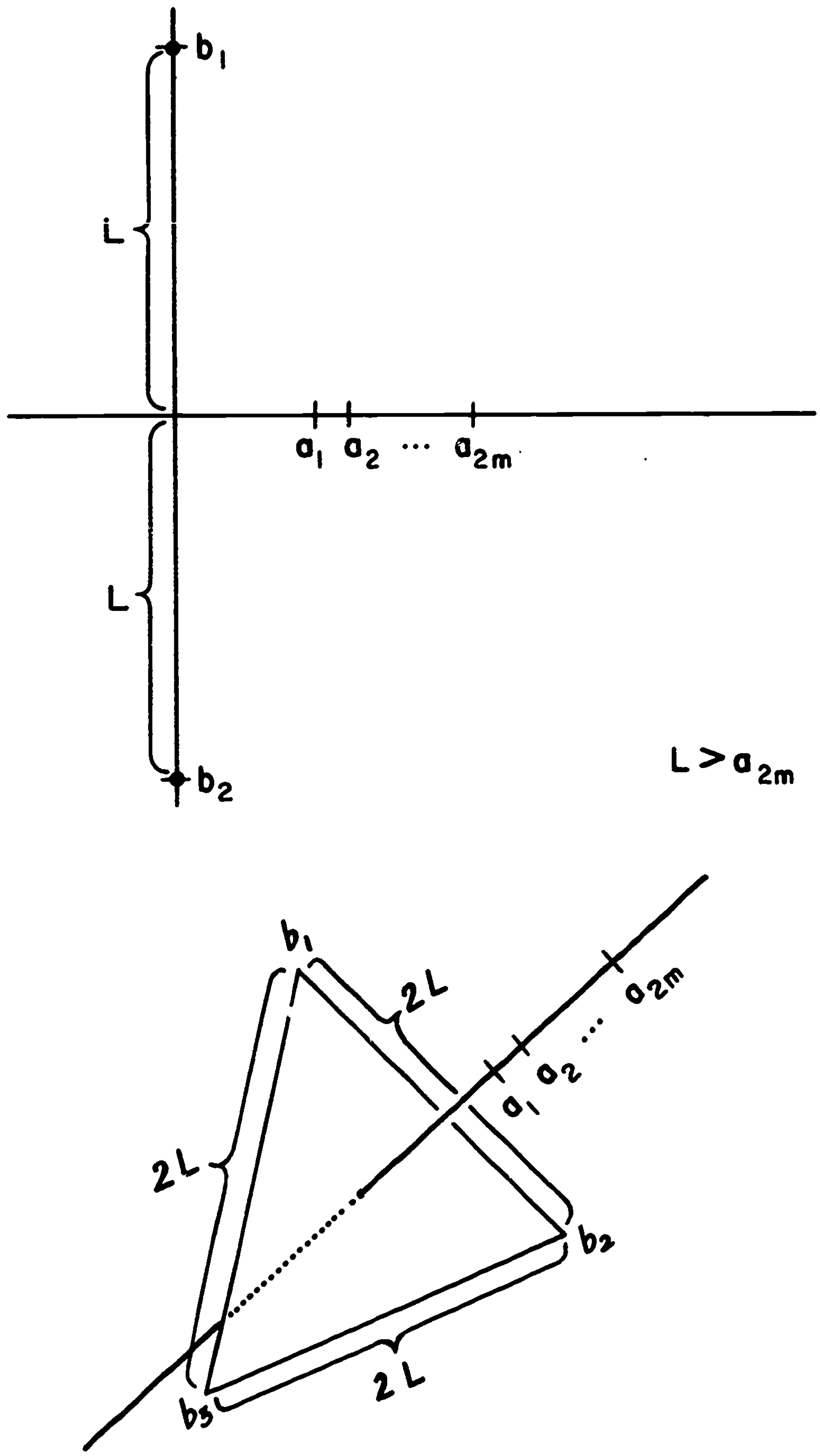

FIGURE 3

The relations between the adjoined points $b_1, \ldots, b_n$ and the points $a_1, \ldots, a_{2m}$, for $n = 2$ and $n = 3$

$\overline{q_i, q_j} = \overline{q_k, q_\ell}$, for $i \neq j$, $k \neq \ell$. Let $x,y,z \in \mathbb{R}^n$ such that, for all $i, j \leq n$, $\overline{q_i, x} = \overline{q_j, x}$, $\overline{q_i, y} = \overline{q_j, y}$, $\overline{q_i, z} = \overline{q_j, z}$. Then x, y, and z are collinear.

Note that beyond $\mathbb{R}^3$ one needs to impose some conditions in addition to $q_i \neq q_j$, $i \neq j$ and $\overline{q_i, x} = \overline{q_j, x}$, $\overline{q_i, y} = \overline{q_j, y}$, $\overline{q_i, z} = \overline{q_j, z}$ in order to force collinearity of x, y, and z. To see this, take $q_1 = \langle 3,0,0,0 \rangle$, $q_2 = \langle 2,0,0,\sqrt{5} \rangle$, $q_3 = \langle 1,0,0,2\sqrt{2} \rangle$, $q_4 = \langle 3/\sqrt{2},0,0,3/\sqrt{2} \rangle$, $x = \langle 0,3,4,0 \rangle$, $y = \langle 0,5,0,0 \rangle$, $z = \langle 0,0,5,0 \rangle$.

Now let $\langle \xi_k \rangle_{k=1}^{\infty}$ be any sequence of elements in $\{-1,1\}$. Let v be any positive real number. Define the pairs $\langle m_k, n_k \rangle$ of complex numbers, for $k = 1, 2, \ldots$ as follows:

$$\langle m_1, n_1 \rangle = \langle 0, v \rangle$$

$$\langle m_2, n_2 \rangle = \langle \frac{v\sqrt{3}}{2}, \frac{v}{2} \rangle$$

$$n_{k+1} = \frac{m_k^2 - n_k^2 + 2n_k m_{k-1} - m_{k-1}^2}{2m_k}$$

$$m_{k+1} = \sqrt{v^2 - (n_{k+1} - m_k)^2}$$

One can prove by induction that m_k and n_k are real numbers.

Define $\alpha_k \in \mathbb{R}^{k+1}$, $k = 0, 1, 2, \ldots$ by $\alpha_0 = \langle 0 \rangle$, $\alpha_1 = \langle m_1, \xi_1 n_1 \rangle$,

$$\alpha_k = \langle 0, \xi_k m_k, \xi_{k-1} n_k, \ldots, \xi_2 n_3, \xi_1 n_2 \rangle$$

Given k, let $\langle r_i \rangle_{i=1}^{k+1}$ be the sequence of $k+1$ points in $\mathbb{R}^{k+1}$ defined by

$$\Pi_j(r_i) = 0 \ , \quad \text{for} \quad j = 1, \ldots, \hbar - i + 1$$
$$\Pi_j(r_i) = \Pi_{i+j-\hbar-1}(\alpha_{i-1}) \ , \quad \text{for} \quad j = \hbar - i + 2, \ldots, \hbar + 1 \ .$$

Lemma 4

Let $v \in \mathbb{R}$, $v > 0$. Let $q_1, \ldots, q_{\hbar+1}$ be points in $\mathbb{R}^{\hbar+1}$ such that $\overline{q_i, q_j} = v$, $i \neq j$. Then there exists a 1-1 function τ on $\mathbb{R}^{\hbar+1}$ into itself such that both τ and its inverse preserve lines and Euclidean distances and there exists a sequence $\langle \xi_i \rangle_{i=1, \ldots, \hbar}$ of elements from $\{-1,1\}$ having the property that, for $r_1, \ldots, r_{\hbar+1}$ as above, $\tau(q_i) = r_i$, $1 \leq i \leq \hbar+1$. Further, if $P \in \mathbb{R}^{\hbar+1}$ and $\overline{P, r_i} = \overline{P, r_1}$, $i = 2, \ldots, \hbar+1$, then $\Pi_j(P) = \xi_{\hbar+2-j}\, \eta_{\hbar+3-j}$, $2 \leq j \leq \hbar+1$.

The only trick involved here is to take τ so that $\Pi_j(\tau(q_i)) = 0$, $1 \leq j \leq \hbar+2-i$. Then, by induction, one can verify that there is a sequence $\langle \xi_i \rangle$ so that $\tau(q_i) = r_i$, $1 \leq i \leq \hbar+1$. Since τ^{-1} preserves lines, it is clear that Lemma 3 follows from Lemma 4.

Lemma 5

For all $n \geq 2$ and, for all $v > 0$, there is a real number $\hat{m}$, $0 < \hat{m} \leq \frac{v\sqrt{3}}{2}$, and there are points $q, q_1, \ldots, q_n \in \mathbb{R}^n$ such that

(i) $\Pi_1(q) = 0$, $\Pi_1(q_i) = 0$, $1 \leq i \leq n$

(ii) $\overline{q_i, q_j} = v$, $i \neq j$

(iii) $\overline{q, q_i} = \overline{q, q_1}$, $1 \leq i \leq n$

(iv) $\overline{q, q_1}^2 = v^2 - \hat{m}^2$

Lemmas 3, 4, and 5 provide us with sufficient machinery for establishing

Lemma 6

Let n, m be positive integers such that m is odd. Then there exists a model $\mathfrak{m} = \langle B, E\rangle$, $|B| = 2m + n$, $E \subseteq B^4$, $\mathfrak{m} \notin K_n$ such that there exists a submodel $\mathfrak{m}_0$ of $\mathfrak{m}$ such that $2m \leq |\mathfrak{m}_0| \leq 2m + n$, $\mathfrak{m}_0 \notin K_n$ and if $\mathfrak{n}$ is a submodel of $\mathfrak{m}_0$ such that $|\mathfrak{n}| = |\mathfrak{m}_0| - 1$, then $\mathfrak{n} \in K_n$.

The proof of Theorem 7 follows immediately from Lemma 6.

Pick any n, m as above; let $A = \{a_1, \ldots, a_{2m}\}$ and D be as in Lemma 2. Choose $b_1, \ldots, b_n$ so that $a_{2m} < b_1 < \ldots < b_n$. Let $B = A \cup \{b_1, \ldots, b_n\}$. Let L be a real number larger than a_{2m}. Apply Lemma 5 with $v = 2L$ and obtain points $q, q_1, \ldots, q_n \in \mathbb{R}^n$, $\hat{m} \in \mathbb{R}$. Choose $\gamma > 0$ so that $\gamma a_{2m} < \hat{m}$. For each $x \in B$, define $x^* \in \mathbb{R}^n$ as follows:

For $1 \leq i \leq 2m$, let $a_i^* = \langle \gamma a_i, \Pi_2(q), \ldots, \Pi_n(q)\rangle$.

For $1 \leq i \leq n$, let $b_i^* - q_i$.

Let $E = D \cup \{\langle x,y,z,w\rangle \in B^4 - A^4 / \overline{x^*, y^*} \leq \overline{z^*, w^*}\}$.

Let $\mathfrak{m} = \langle B, E\rangle$. We shall now verify that $\mathfrak{m}$ satisfies the conditions in the statement of Lemma 6.

Claim 1: $\mathfrak{m} \notin K_n$.

Suppose the contrary. Then there is a homomorphism $f: B \to \mathbb{R}^n$ such that, for all $x, y, z, w \in B$,

$$\langle x,y,z,w\rangle \in E \quad \text{iff} \quad \overline{f(x),\, f(y)} \le \overline{f(z),\, f(w)} \,.$$

For $i = 1, \ldots, n$, let $q_i' = f(b_i)$; and, for $i = 1, \ldots, 2m$, let $a_i' = f(a_i)$. Because $\langle b_i, b_j, b_k, b_\ell\rangle$ and $\langle b_k, b_\ell, b_i, b_j\rangle \in E$, if $i \neq j$, $k \neq \ell$, we know $\overline{q_i', q_j'} = \overline{q_k', q_\ell'}$, $i \neq j$, $k \neq \ell$. Also $\overline{q_i', a_k'} = \overline{q_j', a_k'}$, $1 \le i, j \le n$, $1 \le k \le 2m$. Hence, by Lemma 3, $f(a_1), \ldots, f(a_{2m})$ are collinear. Therefore, there exist points $\langle x_1, \ldots, x_n\rangle$, $\langle y_1, \ldots, y_n\rangle \in \mathbb{R}^n$, and numbers $\alpha_i \in \mathbb{R}$, $1 \le i \le 2m$ such that $f(a_i) = \langle x_1 + \alpha_i y_1, \ldots, x_n + \alpha_i y_n\rangle$. Let $z = \sum_{t=1}^{n} y_t^2$ and define $g: A \to \mathbb{R}$ by $g(a_i) = \alpha_i z$. Then, for all $a_i, a_j, a_k, a_\ell \in A$,

$$\begin{aligned}
\langle a_i, a_j, a_k, a_\ell\rangle \in D \quad &\text{iff} \quad \overline{f(a_i),\, f(a_j)} \le \overline{f(a_k),\, f(a_\ell)} \\
&\text{iff} \quad \sum_{t=1}^{n} y_t^2\, (\alpha_i - \alpha_j)^2 \le \sum_{t=1}^{n} y_t^2\, (\alpha_k - \alpha_\ell) \\
&\text{iff} \quad |g(a_i) - g(a_j)| \le |g(a_k) - g(a_\ell)|
\end{aligned}$$

Therefore g is a homomorphism from $\langle A, D\rangle$ into $\langle \mathbb{R}, \Delta_1\rangle$. Contradiction. Thus we have established Claim 1.

<u>Lemma 7</u>

There exists $\epsilon > 0$ such that if f is any map from A into $\mathbb{R}$ for which $f(a_i) \in [a_i, a_i + \epsilon)$ and if f^* is the map from B into $\mathbb{R}^n$ oftained from f by $f^*(b_i) = q_i$, $1 \le i \le n$, $f^*(a_i) = \langle \gamma f(a_i), \Pi_2(q), \ldots, \Pi_n(q)\rangle$, $1 \le i \le 2m$, then, for all

$\langle x,y,z,w\rangle \in B^4 - A^4$,

$$\overline{x^*,\ y^*} \leq \overline{z^*,\ w^*} \quad \text{iff} \quad \overline{f^*(x),\ f^*(y)} \leq \overline{f^*(z),\ f^*(w)}$$

Proof:

Take $\epsilon < \min (\{|a_i - a_j|/i \neq j\} \cup \{L - a_{2m}\})$

Pick any $\langle x,y,z,w\rangle\ B^4 - A^4$. We shall indicate how one can verify that

$$\overline{x^*,\ y^*} < \overline{z^*,\ w^*} \rightarrow \overline{f^*(x),\ f^*(y)} < \overline{f^*(z),\ f^*(w)}$$

and

$$\overline{x^*,\ y^*} = \overline{z^*,\ w^*} \rightarrow \overline{f^*(x),\ f^*(y)} = \overline{f^*(z),\ f^*(w)}\ .$$

At least one of x,y,z,w is a member of $B - A$. Since $\overline{u,v} = \overline{v,u}$, all $u,v\epsilon\mathbb{R}^n$, we may assume that $\langle x,y,z,w\rangle$ is of one of the following forms:

(i) $a_i,\ a_j,\ b_k,\ b_\ell$

(ii) $a_i,\ a_j,\ a_k,\ b_\ell$

(iii) $a_i,\ b_j,\ a_k,\ a_\ell$

(iv) $a_i,\ b_j,\ a_k,\ b_\ell$

(v) $a_i,\ b_j,\ b_k,\ b_\ell$

(vi) $b_i,\ b_j,\ a_k,\ a_\ell$

(vii) $b_i,\ b_j,\ a_k,\ b_\ell$

(viii) $b_i,\ b_j,\ b_k,\ b_\ell$

Because $\overline{a_i^*,\ a_j^*} < L$, $1 \leq i,\ j \leq 2m$, $L < \overline{a_i^*,\ b_j^*} < 2L$, $1 \leq i \leq 2m,\ 1 \leq j \leq n$, and $\overline{b_i^*,\ b_j^*} = 2L,\ 1 \leq i,\ j \leq n$, all of the above cases other than (iv) may be dealt with quickly and easily. As to (iv), we need only note that, because of our choice of ϵ, everything works out nicely.

$$\overline{a_i^*, b_j^*} < \overline{a_k^*, b_\ell^*} \quad \text{iff} \quad (\gamma a_i)^2 + \overline{q, q_j}^2 < (\gamma a_k)^2 + \overline{q, q_\ell}^2$$

$$\text{iff} \quad a_i < a_k$$

$$\text{iff} \quad f(a_i) < f(a_k)$$

$$\text{iff} \quad (\gamma f(a_i))^2 + \overline{q, q_j}^2 < (\gamma f(a_k))^2 + \overline{q, q_\ell}^2$$

$$\text{iff} \quad \overline{f^*(a_i), f^*(b_j)} < \overline{f^*(a_k), f^*(b_\ell)} .$$

qed.

<u>Claim 2</u>:

For all $i \in \{1, 2, \ldots, 2m\}$, $\mathfrak{M}^{a_i} \in K_n$.

<u>Proof</u>:

Choose $\epsilon > 0$ as in Lemma 7, and apply Lemma 2 to obtain a homomorphism f from $\langle A - \{a_i\}, D_{\restriction A - \{a_i\}}\rangle$ onto a substructure of $\langle \mathbb{R}, \Delta_1\rangle$ st $f(a_j) \in [a_j, a_{j+\epsilon})$. Let f^* be obtained from f as in Lemma 7. Then f^* is a homomorphism from $\mathfrak{M}^{a_i}$ onto a substructure of $\langle \mathbb{R}^n, \Delta_n\rangle$. Pick any $x,y,z,w \in B^4 - \{a_i\}$. If $\langle x,y,z,w\rangle \in B^4 - A^4$, then

$$\langle x,y,z,w\rangle \in E \quad \text{iff} \quad \overline{x^*, y^*} \le \overline{z^*, w^*} \quad \text{iff} \quad \overline{f^*(x), f^*(y)} \le \overline{f^*(z), f^*(w)} .$$

If $\langle x,y,z,w\rangle \epsilon A^4$, then

$$\langle x,y,z,w\rangle \epsilon E \text{ iff } \langle x,y,z,w\rangle \epsilon D \text{ iff } |f(x) - f(y)| \leq |f(z) - f(w)|$$

$$\text{iff } \overline{f^*(x), f^*(y)} \leq \overline{f^*(z), f^*(w)} .$$

<u>Claim 3</u>:

Let $\{c_1, \ldots, c_\ell\}$ and $\{d_1, \ldots, d_\ell\}$ be two sets of distinct elements from $B - A$. Then

$$\mathfrak{m}^{c_1, \ldots, c_\ell} \epsilon K_n \quad \text{iff} \quad \mathfrak{m}^{d_1, \ldots, d_\ell} \epsilon K_n .$$

<u>Proof</u>:

We may assume that there exists k , $0 \leq k < \ell$ so that $c_1 = d_1, \ldots, c_k = d_k$ and $\{c_{k+1}, \ldots, c_\ell\} \cap \{d_{k+1}, \ldots, d_\ell\} = \emptyset$. Because of symmetry we need only show that, if f is a homomorphism from $\mathfrak{m}^{c_1, \ldots, c_\ell}$ onto a substructure of $\langle \mathbb{R}^n, \Delta_n\rangle$, then $\mathfrak{m}^{d_1, \ldots, d_\ell} \epsilon K_n$. Define the function $g:B - \{d_1, \ldots, d_\ell\}$ into $\mathbb{R}^n$ by

$$g(x) = \begin{cases} f(x) , & \text{if } x \epsilon \mathcal{D}(f) \\ f(d_j) , & \text{if } x = c_j, \ k + 1 \leq j \leq \ell . \end{cases}$$

We shall show that g is a homomorphism from $\mathfrak{m}^{d_1, \ldots, d_\ell}$ onto a substructure of $\langle \mathbb{R}^n , \Delta_n\rangle$. Pick any $x,y,z,w \epsilon B - \{d_1, \ldots, d_\ell\}$. First, suppose $\{x,y,z,w\} \cap \{c_1, \ldots, c_\ell\} = \emptyset$.

Then

$$\langle x,y,z,w\rangle \in E \quad \text{iff} \quad \overline{f(x),\ f(y)} \leq \overline{f(z),\ f(w)}$$
$$\text{iff} \quad \overline{g(x),\ g(y)} \leq \overline{g(z),\ g(w)}$$

Now, suppose $\{x,y,z,w\} \cap \{c_1, \ldots, c_\ell\} \neq \emptyset$.

Then $\langle x,y,z,w\rangle \in B^4 - A^4$ and, hence,

$$\langle x,y,z,w\rangle \in E \quad \text{iff} \quad \overline{x^*,\ y^*} \leq \overline{z^*,\ w^*}$$

For $u \in B - \{d_1, \ldots, d_\ell\}$ define u' by

$$u' = \begin{cases} u\ , & \text{if} \quad u \notin \{c_1, \ldots, c_\ell\} \\ d_j, & \text{if} \quad x = c_j,\ k+1 \leq j \leq \ell \end{cases}$$

Then $g(u) = f(u')$.

Because $\overline{c_i^*,\ c_j^*} = \overline{d_i^*,\ d_j^*}$ and, for $t \in B - \{d_1, \ldots, d_\ell, c_1, \ldots, c_\ell\}$,
$\overline{t^*,\ c_i^*} = \overline{t^*,\ d_j^*}$, we know that

$$\overline{x^*,\ y^*} \leq \overline{z^*,\ w^*} \quad \text{iff} \quad \overline{x'^*,\ y'^*} \leq \overline{z'^*,\ w'^*}$$

Hence,

$$\begin{aligned} \langle x,y,z,w\rangle \in E &\leftrightarrow \langle x',\ y',\ z',\ w'\rangle \in E \\ &\leftrightarrow \langle f(x'),\ f(y'),\ f(z'),\ f(w')\rangle \in \Delta_n \\ &\leftrightarrow \langle g(x),\ g(y),\ g(z),\ g(w)\rangle \in \Delta_n\ . \end{aligned}$$

Therefore, g is a homomorphism and $\mathfrak{M}^{d_1, \ldots, d_\ell} \in K_n$. qed.

At this point we are ready to show the existence of the submodel $\mathfrak{M}_0$ of $\mathfrak{M}$. We distinguish three cases as follows:

(1) $\mathfrak{m}^{b_1} \in K_n$

(2) $(\exists \ell)(2 \leq \ell \leq n)\ \mathfrak{m}^{b_1, \ldots, b_{\ell-1}} \notin K_n \wedge \mathfrak{m}^{b_1, \ldots, b_\ell} \in K_n$

(3) $\mathfrak{m}^{b_1, \ldots, b_n} \notin K_n$

In Case (1), we let $\mathfrak{m}_0 = \mathfrak{m}$.
In Case (2), take $\mathfrak{m}_0 = \mathfrak{m}^{b_1, \ldots, b_{\ell-1}}$; and, in
Case (3), take $\mathfrak{m}_0 = \mathfrak{m}^{b_1, \ldots, b_n}$

Then $2m \leq |\mathfrak{m}_0| \leq 2m + n$; and, from Claims 1, 2, 3, it follows that $\mathfrak{m}_0 \notin K_n$; further, if $\mathfrak{n} \in S(\mathfrak{m}_0)$, $|\mathfrak{n}| = |\mathfrak{m}_0| - 1$, then $\mathfrak{n} \in K_n$. This completes the proof of Theorem 7.

The next situation we shall deal with is where measurement-theoretic classes K_n are constructed in terms of a dissimilarity relation generated by another metric in $\mathbb{R}^n$ which has received attention in the literature of mathematical psychology.

2. The "Dominance" Metric

For convenience let us now change some of the referents of some of the symbols used in the previous section of this chapter. Given $x = \langle x_1, \ldots, x_n \rangle$, $y = \langle y_1, \ldots, y_n \rangle \in \mathbb{R}^n$, let

$\overline{x,y} = \max\limits_{1 \leq i \leq n} |x_i - y_i|$. We shall present a proof of

Theorem 8

Let Δ_n be the four-place relation on $\mathbb{R}^n$ given by $xy\Delta_n zw$

iff $\overline{x,y}^n \leq \overline{z,w}^n$. Let K_n be the class of all finite structures belonging to $H^{-1}S\ (\langle \mathbb{R}^n, \Delta_n \rangle)$. Then K_n is not axiomatizable by a universal sentence.

We begin with some combinatorial results.

Lemma 8

Let $n \geq 2$, $L \in \mathbb{R}$, $L > 0$. Let $\mathcal{L}$ be a maximal collection of points $q_i \in \mathbb{R}^n$ st $L \geq \Pi_j(q_i) \geq 0$, $1 \leq j \leq n$, and $\overline{q_i, q_j} = L$, $i \neq j$. Then $|\mathcal{L}| = 2^n$ and

$$\{q_1, \ldots, q_{2^n}\} = \{\langle x_1, \ldots, x_n \rangle \in \mathbb{R}^n / x_i \in \{0,L\},\ 1 \leq i \leq n\}$$

The reader is invited to verify Lemma 8 for the case $n = 2$. We shall show by induction that the lemma holds for all $n \geq 2$. Suppose that the lemma holds for $2 \leq n \leq k$ and that $\mathcal{L}$ is a maximal collection of points in $\mathbb{R}^{k+1}$.

Let

$$A_0 = \{\langle x_1, \ldots, x_{k+1} \rangle \in \mathcal{L} / x_i < L\ ,\ 1 \leq i \leq k+1\}$$
$$A_1 = \{\langle x_1, x_2, \ldots, x_{k+1} \rangle \in \mathcal{L} / x_1 = L\}$$
$$A_2 = \{\langle x_1, x_2, \ldots, x_{k+1} \rangle \in \mathcal{L} / x_1 < L,\ x_2 = L\}$$
$$A_3 = \{\langle x_1, x_2, \ldots, x_{k+1} \rangle \in \mathcal{L} / x_1 < L,\ x_2 < L,\ x_3 = L\}$$
$$\vdots$$
$$A_k = \{\langle x_1, \ldots, x_{k+1} \rangle \in \mathcal{L} / x_1, \ldots, x_{k-1} < L,\ x_k = L\}$$
$$A_{k+1} = \{\langle x_1, \ldots, x_{k+1} \rangle \in \mathcal{L} / x_1, \ldots, x_k < L,\ x_{k+1} = L\}$$

Then $\mathcal{L} = A_0 \cup A_1 \cup \ldots \cup A_{k+1}$. By the induction hypothesis, we know $|A_1| \leq 2^{k}$, $|A_2| \leq 2^{k-1}$; $|A_3| \leq 2^{k-2}$, ..., $|A_k| \leq 2$. Also, $|A_0| \leq 1$ and $|A_{k+1}| \leq 1$. Therefore, $|\mathcal{L}| \leq 2^{k+1}$. Also, since $\mathcal{L}$ is maximal and there is a way to choose elements for each A_i so that $|A_i| = 2^{k-i+1}$, $1 \leq i \leq k+1$ and $|A_0| = 1$, we know that $|\mathcal{L}| = 2^{k+1}$ and the above inequalities are actually equalities. Let

$$\mathcal{L}_j = \{\langle x_1, \ldots, x_j\rangle \in \mathbb{R}^n / x_i \in \{0, L\}, 1 \leq i \leq j\} .$$

By the induction hypothesis, we know that

$$\langle x_1, \ldots, x_{k+1}\rangle \in A_i \rightarrow x_1, \ldots, x_{i-1} < L, x_i = L ,$$

and

$$\langle x_{i+1}, \ldots, x_{k+1}\rangle \in \mathcal{L}_{k+1-i} \quad \text{for} \quad 1 \leq i \leq k+1 .$$

Using the above fact and the fact that if $i \neq j$, $p \in A_i$, $q \in A_j$, then $\overline{p,q} = L$, we may show that, for $\langle x_1, \ldots, x_{k+1}\rangle \in A_i$, where $i \geq 2$, $x_1 = x_2 = \ldots = x_{i-1} = 0$. Then we may verify that

$$A_0 = \{\langle x_1, \ldots, x_{k+1}\rangle / x_1 = x_2 = \ldots = x_{k+1} = 0\} .$$

Therefore,

$$\mathcal{L} = \{\langle x_1, \ldots, x_{k+1}\rangle \in \mathbb{R}^{k+1} / x_i \in \{0, L\}, 1 \leq i \leq k+1\} .$$

qed.

Corollary

Let $q_1, \ldots, q_m$ be any collection of points in $\mathbb{R}^n$ st $\overline{q_i, q_j} = L$, $i \neq j$, and m is maximal. Then $m = 2^n$ and there is a translation $\tau : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\{\tau(q_i)/1 \le i \le m\} = \{\langle x_1, \ldots, x_n\rangle \in \mathbb{R}^n / x_j \in \{0, L\}, 1 \le j \le n\} .$$

Now we are ready to characterize all collections of $2^n - 1$ distinct, pairwise equidistant (by the "dominance" metric) points in $\mathbb{R}^n$. Given $i \in \{1, \ldots, n\}$, let $\hat{}^i$ map $\mathbb{R}^n$ into $\mathbb{R}^{n-1}$ so that if $x = \langle x_1, \ldots, x_n\rangle$, then $\hat{x}^i = \langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\rangle$.

Lemma 9

Let $n \ge 2$, $L \in \mathbb{R}$, $L > 0$. Let $\mathcal{L}$ be a collection of $2^n - 1$ points $q_1, \ldots, q_{2^n-1} \in \mathbb{R}^n$ st $0 \le \Pi_j(q_i) \le L$, $1 \le j \le n$, $1 \le i \le 2^n-1$, and $\overline{q_i, q_j} = L$, $i \ne j$. Then there are numbers i_0, j_0, $1 \le i_0 \le n$, $1 \le j_0 \le 2^n-1$ such that

(i) $\{\hat{q}_1^{i_0}, \ldots, \hat{q}_{2^n-1}^{i_0}\} \subseteq \{\langle x_1, \ldots, x_{n-1}\rangle / x_i \in \{0,L\}, 1 \le i \le n-1\}$

(ii) For $1 \le i \le 2^n-1$, if $i \ne j_0$, then $\Pi_{i_0}(q_i) \in \{0,L\}$

and

(iii) If $p \in \mathbb{R}^n$ and $\Pi_i(p) = \Pi_i(q_{j_0})$, $i \ne i_0$, then $p \in \mathcal{L}$ iff $p = q_{j_0}$.

Proof:

One can verify that Lemma 9 holds for $n = 2$. Now suppose that it holds for all n st $2 \le n \le k$, and let $\mathcal{L}$ be a collection of $2^{k+1}-1$ points in $\mathbb{R}^{k+1}$ satisfying the hypotheses of the lemma. Define sets A_i, $0 \le i \le k+1$ in terms of $\mathcal{L}$ exactly as was done in the proof of Lemma 8. Then, since $|\mathcal{L}| = 2^{k+1} - 1$, there exists h , $0 \leq h \leq k+1$,

st (a) If $0 < i \neq h$, then $|A_i| = 2^{k+1-i}$

(b) If $0 \neq h$, then $|A_0| = 1$

(c) If $h = 0$, then $|A_h| = 0$

(d) If $h \neq 0$, then $|A_h| = 2^{k+1-h} - 1$.

If $h = 0$, it follows from Lemma 8 that

$$\mathcal{L} = \{\langle x_1, \ldots, x_{k+1}\rangle \in \mathbb{R}_n / x_i \in \{0,L\}, 1 \leq i \leq k+1\} - \{\langle 0, 0, \ldots, 0\rangle\}$$

Then we may take $i_0 = k+1$ and j_0 so that $q_{j_0} \in A_{k+1}$. qed.

Thus we may assume that $h \geq 1$. If $h = k+1$, then we may take $i_0 = k+1$ and j_0 so that $q_{j_0} \in A_0$. So we now may assume that $1 \leq h \leq k$. It follows from Lemma 8 and the fact that $\overline{p,q} = L$, for distinct $p, q \in \mathcal{L}$, that, for all $i, j,$ if $i \neq h$, $0 < i < j$, and $r \in A_j$, then $\Pi_i(r) = 0$. Also, for $r \in A_0$ and $i \neq h$, $\Pi_i(r) = 0$. Let $\mathcal{L}_h = \{\langle x_{h+1}, \ldots, x_{k+1}\rangle \in \mathbb{R}^{k+1-h} / (\exists x_1, \ldots, x_h \in \mathbb{R})$ $(\langle x_1, \ldots, x_h, \ldots, x_{k+1}\rangle \in A_h)\}$. By our induction hypothesis, we know what $\mathcal{L}_h$ looks like. Let $\langle y_1, \ldots, y_{k+1-h}\rangle$ be the point of $\mathbb{R}^{k+1-h}$ corresponding to the q_{j_0} obtained when Lemma 9 is applied to $\mathcal{L}_h$. Let i_0' be the index $1 \leq i_0' \leq k+1-h$ corresponding to i_0 in the lemma. If $y_{i_0'} \notin \{0,L\}$, then it is easy to verify that one can satisfy the conclusion of Lemma 9 (for $\mathbb{R}^{k+1}$) by choosing $i_0 = h + i_0'$ and j_0 so that $q_{j_0} \in A_h$ and $\langle \Pi_{h+1}(q_{j_0}), \ldots, \Pi_{k+1}(q_{j_0})\rangle = \langle y_1, \ldots, y_{k+1-h}\rangle$. We now need only deal with the case where $y_{i_0} \in \{0,L\}$. Therefore, by the induction hypothesis, we know that there is some element

$\langle p_1, \ldots, p_{k+1-h} \rangle \in \{0,L\}^{k+1-h}$ such that

$\mathcal{L}_h = \{0,L\}^{k+1-h} - \{\langle p_1, \ldots, p_{k+1-h} \rangle\}$.

Case 1: $\langle p_1, \ldots, p_{k+1-h} \rangle \notin \{0\}^{k+1-h}$

Let θ be the least positive integer such that $p_\theta = L$. Then one can show that, for $i \in \{0, 1, \ldots, k+1\} - \{h, h+\theta\}$ if $q \in A_i$, then $\Pi_h(q) = 0$. Also, for all $q \in A_{h+\theta}$ for which it is not the case that $\langle \Pi_{h+1}(q), \ldots, \Pi_{k+1}(q) \rangle = \langle p_1, \ldots, p_{k+1-h} \rangle$ one can show that $\Pi_h(q) = 0$. Therefore, we may satisfy the conclusion of Lemma 9 by choosing $i_0 = h$ and j_0 so that q_{j_0} is the member of $A_{h+\theta}$ for which $\langle \Pi_{h+1}(q_{j_0}), \ldots, \Pi_{k+1}(q_{j_0}) \rangle = \langle p_1, \ldots, p_{k+1-h} \rangle$.

qed.

Case 2: $\langle p_1, \ldots, p_{k+1-h} \rangle \in \{0\}^{k+1-h}$

In this case one should choose $i_0 = h$ and j_0 so that $q_{j_0} \in A_0$. Then one can verify the conclusion of Lemma 9.

Our next goal is to prove a collinearity result like Lemma 3 only for the "dominance" metric.

Lemma 10

Let $L \in \mathbb{R}$, $L > 0$, $n \geq 2$.

Let $q_1, \ldots, q_{2^n-1} \in \mathbb{R}^n$ st $\overline{q_i, q_j} = L$, $i \neq j$. Let $x,y,z \in \mathbb{R}^n$

st $\overline{x,q_i} = \overline{x,q_1}$, $\overline{y,q_i} = \overline{y,q_1}$, $\overline{z,q_i} = \overline{z,q_1}$, $1 \leq i \leq 2^n-1$, and $\overline{x,q_1} < L$, $\overline{y,q_1} < L$, $\overline{z,q_1} < L$. Then x, y, and z are collinear.

Proof: Let $\mathcal{L} = \{q_1, \ldots, q_{2^n-1}\}$.

Let $\mathcal{K} = \{x \in \mathbb{R}^n / \overline{x,q_i} = \overline{x,q_1}$ and $\overline{x,q_1} < L\}$. We need to show that if $x,y,z \in \mathcal{K}$, then x,y, and z are collinear. Because translations preserve lines, we may assume that $0 \leq \Pi_j(q_i) \leq L$, for $1 \leq i \leq 2^n-1$ and $1 \leq j \leq n$. Hence, Lemma 9 is applicable. Moreover, we may assume that $i_0 = n$. Let $e = \Pi_{i_0}(q_{j_0})$. Note that, for all $\langle x_1, \ldots, x_n \rangle \in \mathcal{K}$, $0 < x_i < L$, $1 \leq i \leq n$. Let $\mathcal{K}' = \{x \in \mathcal{K} / \overline{x,q_1} = L/2\}$, $\mathcal{K}'' = \mathcal{K} - \mathcal{K}'$.

Claim 1: $\mathcal{K}' = \{L/2\}^n$

By Lemma 9, we know that, for all i , $1 \leq i \leq n$, there exist j, j' , $1 \leq j , j' \leq 2^n-1$ st $\Pi_i(q_j) = 0$ and $\Pi_i(q_{j'}) = L$. Because of this, one can show that, for all $\langle x_1, \ldots, x_n \rangle \in \mathcal{K}'$, $x_i = \frac{L}{2}$, $1 \leq i \leq n$. It is also straightforward to verify that $\{L/2\}^n \subseteq \mathcal{K}'$.

qed.

Now, for $1 \leq i \leq n-1$, let c_i map $\mathcal{K}''$ into $\{0,L\}$ as follows, given $x = \langle x_1, \ldots, x_n \rangle \in \mathcal{K}''$.

$$c_i(x) = \begin{cases} 0 , & \text{if } x_i \neq \overline{x,q_1} \\ L , & \text{otherwise} \end{cases}$$

Claim 2: For all $x,y \in \mathcal{K}''$ and $i \in \{1, \ldots, n-1\}$, $c_i(x) = c_i(y)$.

We shall show that, for all $x \in \mathcal{K}''$, $\langle c_1(x), \ldots, c_{n-1}(x), e\rangle \in \mathcal{L}$. Suppose the contrary. Then there exists

$x \in \mathcal{K}''$ st $p(x) = \langle c_1(x), \ldots, c_{n-1}(x), 0\rangle \in \mathcal{L}$ and

$$q(x) = \langle c_1(x), \ldots, c_{n-1}(x), L\rangle \in \mathcal{L} .$$

Therefore, $\overline{x,q_1} = \overline{x,p(x)} \geq |x_i - c_i(x)| = \begin{cases} x_i & , \text{ if } x_i \neq \overline{x,q_1} \\ L - x_i, & \text{otherwise} , \end{cases}$

for all $1 \leq i \leq n-1$.

From this one can show that $|x_i - c_i(x)| < \overline{x,q_1}$,

for $1 \leq i \leq n-1$.

Therefore, since $\overline{x,p(x)} = \overline{x,q(x)} = \overline{x,q_1}$, $x_n = \overline{x,q_1}$ and $L - x_n = \overline{x,q_1}$. Hence, $\overline{x,q_1} = \frac{L}{2}$, which is a contradiction, because $x \in \mathcal{K}''$. Now pick any $x,y \in \mathcal{K}''$. By the above we know that $\langle c_1(x), \ldots, c_{n-1}(x), e\rangle = q_{j_0} = \langle c_1(y), \ldots, c_{n-1}(y), e\rangle$, so $c_i(x) = c_i(y)$, $1 \leq i \leq n-1$. qed.

Claim 3: If $\mathcal{K}'' \neq \emptyset$, then $e \in \{0,L\}$.

Suppose there is an element $x \in \mathcal{K}''$. Let $p(x) = \langle c_1(x), \ldots, c_{n-1}(x), e\rangle \in \mathcal{L}$. As above, $|x_i - c_i(x)| < \overline{x,q_1}$, for $1 \leq i \leq n-1$. Therefore, $|x_n - e| = \overline{x,q_1}$, from which it follows that $e \in \{0,L\}$. qed.

Note that if $\mathcal{K}'' = \emptyset$, then our lemma follows immediately from Claim 1; thus we may assume that $\mathcal{K}'' \neq \emptyset$. By virtue of Claim 2, we

may write 'c_i' to indicate the single member of $\{c_i(x)/x\epsilon\mathcal{K}''\}$.
Because of Claim 3 and Lemma 9, we may distinguish the following two cases:

$$\text{(I)} \quad \langle c_1, \ldots, c_{n-1}, 0\rangle \epsilon \mathcal{L}$$
$$\langle c_1, \ldots, c_{n-1}, L\rangle \notin \mathcal{L}$$
$$\text{(II)} \quad \langle c_1, \ldots, c_{n-1}, 0\rangle \notin \mathcal{L}$$
$$\langle c_1, \ldots, c_{n-1}, L\rangle \epsilon \mathcal{L}$$

(I): Define $c_n = L$.

Let $\mathcal{H} = \{\langle L-c_1, \ldots, L-c_n\rangle + \lambda \langle \frac{2c_1-L}{L}, \ldots, \frac{2c_n-L}{L}\rangle / 0 \leq \lambda \leq L\}$.
We shall show that $\mathcal{K}'' \subseteq \mathcal{H}$. Pick any $x = \langle x_1, \ldots, x_n\rangle \epsilon \mathcal{K}''$.
Define $\tilde{c}_i = L - c_i$.

For $i < n$, consider $\langle c_1, \ldots, \tilde{c}_i, \ldots, c_{n-1}, L\rangle \epsilon \mathcal{L}$. Since $\langle c_1, \ldots, c_{n-1}, 0\rangle \epsilon \mathcal{L}$, $x_n = \overline{x,q_1} \neq L/2$. Hence, $|L - x_n| \neq \overline{x,q_1}$; and, therefore, $|x_i - \tilde{c}_i| = \overline{x,q_1}$.

Let $\lambda = \overline{x,q_1}$. Then $x_i = \begin{cases} \lambda & , \text{ if } c_i = L \\ L - \lambda, & \text{if } c_i = 0 \end{cases}$, $1 \leq i \leq n-1$ and $x_n = \lambda$. Therefore,

$$x = \langle L-c_1, \ldots, L-c_n\rangle + \lambda \langle \frac{2c_1-L}{L}, \ldots, \frac{2c_n-L}{L}\rangle \epsilon \mathcal{H} ,$$

So $\mathcal{K}'' \subseteq \mathcal{H}$. qed.

Note that $\mathcal{K}' \subseteq \mathcal{H}$, because we may choose $\lambda = \frac{L}{2}$.
Therefore $\mathcal{K} \subseteq \mathcal{H}$.

In Case (II) one can similarly verify that all members of $\mathcal{K}$

are collinear. Therefore, Lemma 10 holds.

The following trivial lemma will be useful for reference.

Lemma 11

Let $L \in \mathbb{R}$, $L > 0$. Let $\{q_1, \ldots, q_{2^n-1}\} = \{0, 2L\}^n - \{0\}^n$.

Then

(i) $0 \leq \Pi_i(q_j) \leq 2L$, $1 \leq i \leq n$, $1 \leq j \leq 2^n-1$.

(ii) $\overline{q_i, q_j} = 2L$, $i \neq j$

(iii) For all $x \in \mathbb{R}$ st $0 < x < L$, if $p \in \mathbb{R}^n$ is given by $\Pi_i(p) = x$, $1 \leq i \leq n$, then, for all $1 \leq i, j \leq 2^n-1$, $\overline{q_i, p} = \overline{q_j, p}$ and $L < \overline{q_i, p} < 2L$.

Let Lemma 12 be the lemma whose statement consists of exactly the same symbols used in the statement of Lemma 6. Theorem 8 is an immediate consequence of Lemma 12, which we shall now prove.

Pick any n, m as in the statement of Lemma 12. Let $A = \{a_1, \ldots, a_{2m}\}$ and D be as in Lemma 2. Choose $b_1, \ldots, b_{2^n-1}$ st $a_{2m} < b_1 < \ldots < b_{2^n-1}$. Let $B = A \cup \{b_1, \ldots, b_{2^n-1}\}$. Let L be be a real number larger than a_{2m}. Let $q_1, \ldots, q_{2^n-1}$ be as in Lemma 11. For each $x \in B$, define $x^* \in \mathbb{R}^n$ by

$$x^* = \begin{cases} \langle a_i, \ldots, a_i \rangle, & \text{if } x = a_i \\ q_j, & \text{if } x = b_j \end{cases}$$

Let $E = D \cup \{\langle x,y,z,w \rangle \in B^4 - A^4 / \overline{x^*, y^*} \leq \overline{z^*, w^*}\}$. Let $\mathfrak{M} = \langle B, E \rangle$. At this point it should surprise nobody that the obvious analogues of

Lemma 7 and Claims 1, 2, 3 in the proof of Lemma 6 may all be established. Therefore, Theorem 8 holds.

Let us finish this chapter by noting that not every metric in $\mathbb{R}^n$ leads to classes K_n which fail to be universally axiomatizable. For example, let ρ be the metric in $\mathbb{R}^n$ given by

$$\rho(x,y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y . \end{cases}$$

Let K_n be the class of all finite members of $H^{-1}S(\mathbb{R}^n, \Delta_n)$, where $xy\Delta_n zw$ iff $\rho(x,y) \leq \rho(z,w)$, for $x,y,z,w \in \mathbb{R}^n$. One can easily see that K_n is universally axiomatized by the following axioms in a language containing the four-place relation symbol D :

(i) $abDaa$ is an equivalence relation.

(ii) $abDcd \leftrightarrow [abDaa \vee \neg(cdDcc)]$.

APPENDIX

The theorems presented in the main body of this thesis have shown that it is impossible to achieve various representation results with a finite number of necessary universal axioms. The theorem to follow establishes the impossibility of obtaining a combined representation-uniqueness result by using a specific necessary non-universal axiom, τ , along with a finite number of necessary universal axioms.

<u>Theorem</u>

Let Δ be the relation $x - y \leq z - w$ on the real numbers. Let K be the class of all finite $\mathfrak{M} = \langle A, D \rangle$, $D \subseteq A^4$, such that $\mathfrak{M}$ is embeddable in $\langle \mathbb{R}, \Delta \rangle$ by a homomorphism which is unique up to a linear transformation.

(1) K is not universally axiomatizable, because $S(K) \nsubseteq K$.

(2) Let $\equiv$ be the binary relation $(abDaa \wedge aaDab)$ on A . Let τ be the following sentence:

$$(\exists a,b,c \in A)(a \not\equiv b \wedge a \not\equiv c \wedge b \not\equiv c) \rightarrow (\exists a,b,c,d \in A)$$
$$(a \not\equiv b \wedge a \not\equiv c \wedge abDcd \wedge cdDab) .$$

Then, for all $\mathfrak{M} \in K$, τ tr $\mathfrak{M}$.

(3) There is no universal sentence σ such that, for all finite $\mathfrak{M}$,

$\mathfrak{M} \in K$ iff $(\sigma \wedge \tau)$ tr $\mathfrak{M}$.

Part (1) follows from (2), which may be established by showing that if τ fails to hold in a structure $\mathfrak{M} \in K$, then one can violate the uniqueness condition for membership in K by defining homomorphisms similar to those involved in Lemma 2. Part (3) may be proved by using Theorems 15, 16 on p. 39 of Suppes-Zinnes [13]. These theorems enable one to show that if $\mathfrak{M}$ is a 2m-element model as constructed in the proof of Theorem 3, then, for every $a \in \mathcal{D}(\mathfrak{M})$, there exists an extension $\mathfrak{N}$ of $\mathfrak{M}^a$ st $\mathfrak{N} \in K$.

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